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Tensor Algebra, Linear Algebra, Multilinear Algebra

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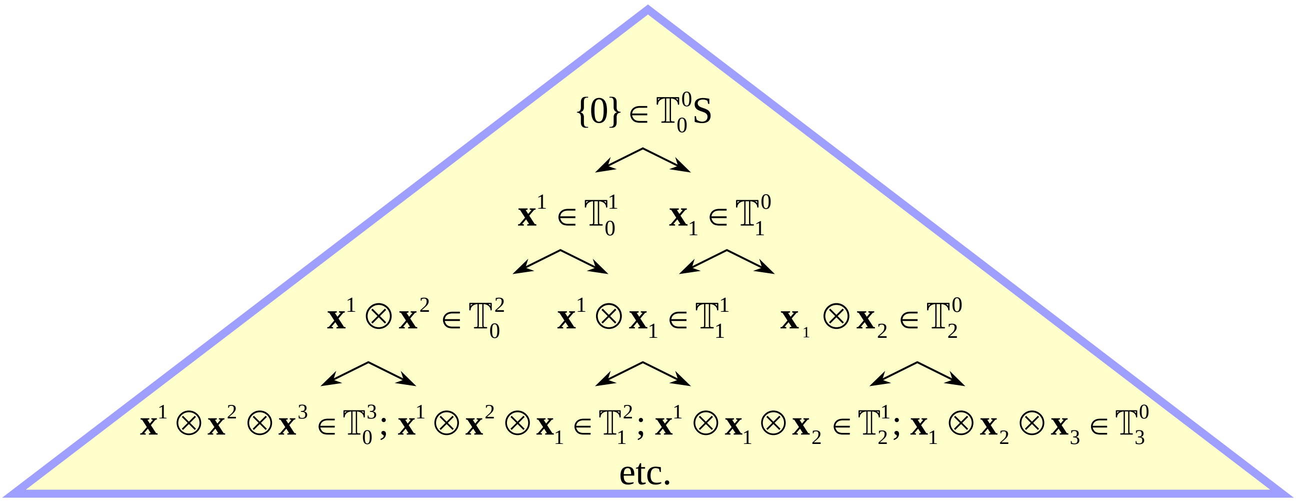


Preface

These notes on tensor algebra, linear algebra as well as multilinear algebra have been prepared as an easy reference for my students attending my lectures or university courses in

Physical Geodesy and Geometric Geodesy
Satellite Geodesy
Differential Geometry and Map Projections.

This is not a whole lot, and in this condensed form would occupy perhaps only a small booklet. My intention is allocating various topics from the algebra of tensors, both linear and multilinear, as following:

> At first we want to transfer the idea that tensors as they appear in all sciences are *not* just matrices. They are subject to a certain algebra. For instance, the 2-tensor as an element of the space of bilinear functions is represented in a bilinear basis. In this bilinear basis the 2-tensor has *coordinates* which are collected in a *two-dimensional array*. Such a two-dimensional array is conventionally called "matrix" (with special reference to *Asterix and Obelix*), a notion introduced by *A. Cayley*. In contrast, the 3-tensor as an element of the space of trilinear functions is represented in a trilinear basis. Again in this trilinear basis the 3-tensor has *coordinates*. Those coordinates of a 3-tensor are collected in a *three-dimensional array* subject to *array algebra*. But the classical matrix algebra fails to identify three-dimensional arrays of real numbers, complex numbers, quaternions (Hamiltonians) or octonians.

As a reference accompanying various lecture series the text is rather advanced. Our aim, however, was not a most lively presentation of ideas involved, but rather a review with special emphasis on other textbooks. For any page of our booklet about ten textbooks are available on the special subject of that page. Instead we have focused on presenting various "useful" algebras. All related examples were given in those courses we quoted earlier.

At first we believed to attach various parts of this booklet to other special courses we already referred to. But over the years those courses at various universities, both on the undergraduate and graduate level, we learnt these lecture notes take away *too much space and time*. With these experiences we decided to present to you this “special” booklet.

Perhaps you, the potential reader, become more interested if we *open the door* to the room of subjects treated under tensor algebra, linear algebra as well as multi-linear algebra. Indeed with a historical reference to

Mûsâ al - Khowârizimi

we outline “al jabr”, namely what *Nicholas Bourbabi* and his disciples called algebra.

> From the eleventh century on, European scholars began to visit Islamic mathematicians to learn about the new numerals. Abu Jafar Muhammad ibn Mûsâ al - Khowârzimi - Muhammad, father of Jafar, son of Mûsâ, the Khowârzimian (680-750) – Khowârzimian is the old Persia - had written a treatise on Arabic numerals which survives in the form of a Latin translation dating from the twelfth century. A copy of this was found in 1857 in the library of Cambridge University. This book was the major vehicle by which the gobar Arabic place system entered European civilization. The Latin form of Khowârzimi gave us the word “*algorithm*”. Another book by him, *ilm al-jabr wa'l-muqabalah* (*The science of reduction and equation*) gave us *algebra.*

To your surprise, perhaps, we begin with multilinear algebra. **§1** introduces the *p*- contravariant, *q*-covariant tensor space or space of multilinear functions. Special emphasis is on

$$\otimes \ldots \otimes$$

which as “*opera*” identifies the *tensor product*. We immediately jump into the fundamental decomposition of the space of the multilinear functions into the subspaces of type

- symmetric multilinear functions,
- antisymmetric multilinear functions,
- residual multilinear functions.

Various examples are given in *Boxes*. “Hand-in-hand” with this decomposition goes the introduction of

the interior product	and	the exterior product
$\vee \ldots \vee$		$\wedge \ldots \wedge$

also called "*wedge product*" or "*skew product*". We shortly refer to "*array algebra*" and "*matrix algebra*" being related to the coordinates of multilinear functions, the 2- contravariant, 0-covariant tensor, for instance. Special focus is on the *Hodge dualizer* or

$$*$$

the *Hodge star operator* within the algebra A_p^q of antisymmetric multilinear functions. We conquer the wonderful world of the basis and the associated cobasis of antisymmetric multilinear functions. **§2** brings us back to linear algebra. We enjoy opera "*join*" and opera "*meet*", "Ass", "Uni", "Comm", the *ring* with identity, anticommutativity, namely

- division algebra
- non-associative algebra
- Lie algebra, Killing analysis
- Witt operator algebra
- Boole algebra
- composition algebra

Various composition algebras equipped with an additional structure, the topological structure of type scalar product, norm or metric are considered:

- **matrix algebra as division algebra**
  (*Cayley inverse*)
- **complex algebra as a division algebra as well as a composition algebra**
  (*Clifford algebra Cl* (0;1))
- **quaternion algebra as a division algebra as well as a composition algebra**
  (*Clifford algebra Cl* (0;2))
- **the letter of W. R. Hamilton to his son**
  (16th October 1843)
- **octonion algebra as a non-associative algebra as well as a composition algebra**
  (Clifford algebra with respect to $\mathbb{H} \times \mathbb{H}$)

§3 is an *intermezzo* to classify antisymmetric and symmetric tensor-valued functions. Of special importance is the decomposition of an antisymmetric multilinear functions into p -vectors, also called "*blades*" which takes up a lot of space. The treatment of orthogonal *Clifford algebra Cl* (p, q) in **§4** is the highlight of our booklet. Here we refer to the *Clifford product*

$$\overset{*}{\wedge} \dots \overset{*}{\wedge}$$

which is between the interior product ("dot product", scalar product) and the exterior product ("cross product", "wedge product") generating *Clifford numbers*. In particular, we highlight the cyclic structure ("*chess board*") as well as graded algebras (cyclic group). The related examples document the power of this great algebra in the applied sciences.

From some point of view the various algebras presented here are rather old fashioned. Indeed we did not include *von Neumann* algebras which play a key role in *quantum statistics* or *non commutative algebras*, also called *super algebras*, which are key elements of *quantum mechanics*, *quantum gravity* and *quantum electrodynamics*. Instead we give some references on *von Neumann algebras* like [References] and on *non commutative* algebras or *super algebras* like *Constantinescu, F.* and *de Groote, H. F.* (1994), [References]

In contrast, we want to promote *Clifford algebra* which is more or less unknown in the applied sciences different from mathematics. For a more detailed introduction into *Clifford algebra and its fascinating chess board* as well as *Clifford analysis* let us refer to [References]. Please, accept our advertisement for the yearly international conference on

> Clifford Algebras and their applications in Mathematical Physics
> (http://clifford.physik.uni-konstanz.de/ fanser/CL)

to be held at various countries. The topics are

- **Clifford algebra and analysis**

Dirac operators, wavelets, nonlinear transformations, harmonic analysis, Fourier analysis, singular integral operators, discrete potential theory, initial value problems, boundary value problems;

- **Geometry**

Differential geometry, geometric index theory, non commutative geometry, spectral triplets, reconstruction theorem, geometric integral transforms, spin structures and Dirac operator, K-theory, projective geometry and twistor, *Seiberg-Witten* theory, quaternionic geometry;

- **Mathematical structures**

Hopf algebras and quantum groups, category theory, structured methods, quadratic forms, *Hermitean forms*, *Witt-groups*, *Clifford algebras* over arbitrary fields, *Lie algebras*, spinor representations, *exceptional Lie algebras*, *Super Lie algebras*, *Clifford algebras and their generalizations*, infinite dimensional *Clifford algebras* and *Clifford bundles*;

- **Physics**

Perturbative renormalization and *Hopf algebra* antipodes, spectral triplets, elementary particle physics, *q*-deformations, noncommutative space-time, quantum field theory using *Hopf algebras*, spin foams, quantum gravity, quaternionic quantum mechanics and quantum fields, *Dirac equations* in electronic physics, electrodynamics, non-associative structures, octonians, division algebras and their applications in physics;

- **Applications in computer science, robotics, engineering**

quantum computers, error corrections, algorithms, robotics, space control, *navigation*, cybernetics, image processing and engineering, neural networks.

Contents

The key word "*algebra*" is derived from the name and the work of the ninth-century scientist Mohammed ibn *Mûsâ al Khowârzimian* who was born in what is now Uzbekistan and worked in Bagdad at the court of *Harun al-Rashid's son*. "*al-jabr*" appears in the title of his book *Kitab al-jabr wail muqabala* where he discusses symbolic methods for the solution of equations (*F. Rosen*: The algebra of Mohammed Ben Musa, London: Oriental Translation Fund 1831, *K. Vogel*: Mohammed Ibn Musa Alchwarizmi's Algorismus; Das fruehste Lehrbuch zum Rechnen mit indischen Ziffern, 1461, Otto Zeller Verlagsbuchhandlung, Aalen 1963). Accordingly what is an *algebra* and how is it tied to the notion of a *vector space*, a *tensor space*, respectively ? By an *algebra* we mean a *set* $\mathbb{S}$ *of elements* and a *finite set* ***M*** *of operations*. Each *operation* $(\text{opera})_k$ is a single-valued function assigning to every finite ordered sequence $(x_1,...,x_n)$ of $n = n(k)$ elements of $\mathbb{S}$ *a value* $(\text{opera})_k$ $(x_1,...,x_k) = x_1$ in $\mathbb{S}$. In particular for $(\text{opera})_k$ (x_1, x_2) the operation is called binary, for $(\text{opera})_k$ (x_1, x_2, x_3) ternary, in general for $(\text{opera})_k$ $(x_1,...,x_n)$ *n*-array. For a given set of operation symbols $(\text{opera})_1$, $(\text{opera})_2$, … , $(\text{opera})_k$ we define a *word*. In *linear algebra* the set ***M*** has basically two elements, namely two internal relations $(\text{opera})_1$ *worded* "addition" (including inverse addition: subtraction) and $(\text{opera})_2$ *worded* "multiplication" (including inverse multiplication: division). *Here* the elements of the set $\mathbb{S}$ are *vectors* over the field $\mathbb{R}$ of real numbers as long as we refer to *linear algebra*. In contrast, in *multilinear algebra* the elements of the set $\mathbb{S}$ are *tensors* over the field of *real numbers* $\mathbb{R}$. Only later *modules* as generalizations of vectors of linear algebra are introduced in which the "scalars" are allowed to be from *an arbitrary ring* rather than the field $\mathbb{R}$ of *real numbers*.

Chapter 1

Tensor Algebra

Let us assume that you as a potential reader are in some way familiar with the elementary notion of a three-dimensional vector space $\mathbb{X}$ with elements called vectors $\mathbf{x} \in \mathbb{R}^3$, namely the intuitive space "we locally live in". Such an elementary vector space $\mathbb{X}$ is equipped with a metric to be referred to *as three-dimensional Euclidean.* As a three-dimensional vector space we are going to give it a *linear and multilinear algebraic structure.* In the context of structure mathematics based upon

(i) order structure
(ii) topological structure
(iii) algebraic structure

an algebra is constituted if at least two relations are established, namely one *internal* and one *external*. We start with *multilinear algebra*, in particular with the multilinearity of the tensor product before we go back to *linear algebra*, in particular to *Clifford algebra.*

1-1 Multilinear functions and the tensor space $\mathbb{T}_q^p$

Let $\mathbb{X}$ be a finite dimensional linear space, e. g. a vector space over the field $\mathbb{R}$ of *real numbers*, in addition denote by $\mathbb{X}^*$ its dual space such that $n = \dim \mathbb{X} = \dim \mathbb{X}^*$**.** Complex, quaternion and octonian numbers $\mathbb{C}$, $\mathbb{H}$ and ***O*** as well as *rings* will only be introduced later in the context. For $p, q \in \mathbb{Z}^+$ being an element of positive integer numbers we introduce

$$\mathbb{T}_q^p(\mathbb{X}, \mathbb{X}^*)$$

as the *p*-contravariant, *q*-covariant-*tensor space or space of multilinear functions*

$$f : \mathbb{X}^* \times ... \times \mathbb{X}^* \times ... \times \mathbb{X} \to \mathbb{R}^{p \dim \mathbb{X}^* \times q \dim \mathbb{X}}$$

If we assume $\mathbf{x}^1, ..., \mathbf{x}^p \in \mathbb{X}^*$ and $\mathbf{x}_1, ..., \mathbf{x}_p \in \mathbb{X}$, then

$$\boxed{\mathbf{x}^1 \otimes ... \otimes \mathbf{x}^p \otimes \mathbf{x}_1 \otimes ... \otimes \mathbf{x}_q \in \mathbb{T}_q^p(\mathbb{X}^*, \mathbb{X})}$$

holds. *Multilinearity is understood as linearity in each variable.* "$\otimes$" identifies the tensor product, *the Cartesian product* of elements $(\mathbf{x}^1, ..., \mathbf{x}^p, \mathbf{x}_1, ..., \mathbf{x}_q)$

Example 1-1: Bilinearity of the tensor product $\mathbf{x}_1 \otimes \mathbf{x}_2$

For every $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{X}$, $\mathbf{x}, \mathbf{y} \in \mathbb{X}$ and $r \in \mathbb{R}$ bilinearity implies

$(\mathbf{x}+\mathbf{y}) \otimes \mathbf{x}_2 = \mathbf{x} \otimes \mathbf{x}_2 + \mathbf{y} \otimes \mathbf{x}_2$ *(internal left- linearity)*

$\mathbf{x}_1 \otimes (\mathbf{x}+\mathbf{y}) = \mathbf{x}_1 \otimes \mathbf{x} + \mathbf{x}_1 \otimes \mathbf{y}$ *(internal right- linearity)*

$r\mathbf{x} \otimes \mathbf{x}_2 = r(\mathbf{x} \otimes \mathbf{x}_2)$ *(external left- linearity)*

$\mathbf{x}_1 \otimes r\mathbf{y} = r(\mathbf{x}_1 \otimes \mathbf{y})$ *(external right- linearity)* ♣

The generalization of bilinearity of $\mathbf{x}_1 \otimes \mathbf{x}_2 \in \mathbb{T}_2^0$ to *multilinearity* of

$$\mathbf{x}^1 \otimes \ldots \otimes \mathbf{x}^p \otimes \mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_p \in \mathbb{T}_q^p$$

is obvious.

Definition 1-1 (multilinearity of tensor space $\mathbb{T}_q^p$):

For every $\mathbf{x}^1, \ldots, \mathbf{x}^p \in \mathbb{X}^*$ *and* $\mathbf{x}_1, \ldots, \mathbf{x}_q \in \mathbb{X}$ *as well as* $\mathbf{u}, \mathbf{v} \in \mathbb{X}^*$, $\mathbf{x}, \mathbf{y} \in \mathbb{X}$ *and* $r \in \mathbb{R}$ *multilinearity implies*

$$(\mathbf{u}+\mathbf{v}) \otimes \mathbf{x}^2 \otimes \ldots \otimes \mathbf{x}^p \otimes \mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_q =$$
$$= \mathbf{u} \otimes \mathbf{x}^2 \otimes \ldots \otimes \mathbf{x}^p \otimes \mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_q + \mathbf{v} \otimes \mathbf{x}^2 \otimes \ldots \otimes \mathbf{x}^p \otimes \mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_q$$

(internal left – linearity)

$$\mathbf{x}^1 \otimes \ldots \otimes \mathbf{x}^p \otimes (\mathbf{x}+\mathbf{y}) \otimes \mathbf{x}_2 \otimes \ldots \mathbf{x}_q =$$
$$= \mathbf{x}^1 \otimes \ldots \otimes \mathbf{x}^p \otimes \mathbf{x} \otimes \mathbf{x}_2 \otimes \ldots \otimes \mathbf{x}_q + \mathbf{x}^1 \otimes \ldots \otimes \mathbf{x}^p \otimes \mathbf{y} \otimes \mathbf{x}_2 \otimes \ldots \otimes \mathbf{x}_q$$

(internal right – linearity)

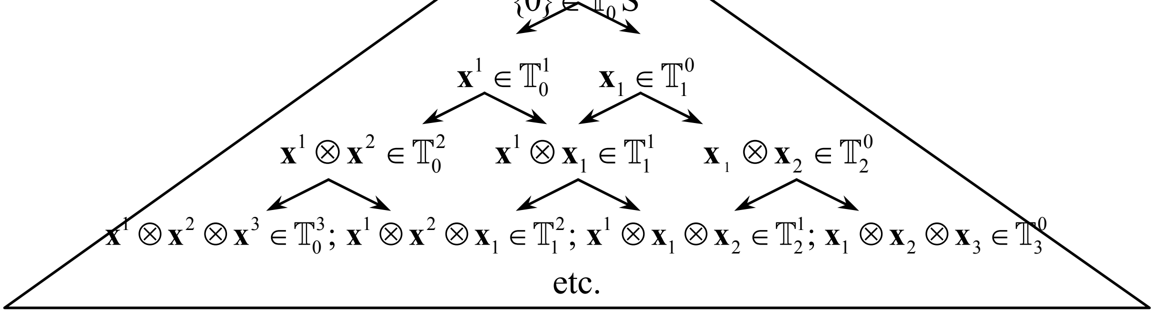
$$r\,\mathbf{u} \otimes \mathbf{x}^2 \otimes \ldots \otimes \mathbf{x}^p \otimes \mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_q = r(\mathbf{u} \otimes \mathbf{x}^2 \otimes \ldots \otimes \mathbf{x}^p \otimes \mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_q)$$

(external left – linearity)

$$\mathbf{x}^1 \otimes \ldots \otimes \mathbf{x}^p \otimes r\,\mathbf{x} \otimes \mathbf{x}_2 \otimes \ldots \otimes \mathbf{x}_q = r(\mathbf{x}^1 \otimes \ldots \otimes \mathbf{x}^p \otimes \mathbf{x} \otimes \mathbf{x}_2 \otimes \ldots \otimes \mathbf{x}_q)$$

(external right – linearity).

A possible way to visualize the different multilinear functions which span $\{\mathbb{T}_0^0, \mathbb{T}_0^1, \mathbb{T}_1^0, \mathbb{T}_0^2, \mathbb{T}_1^1, \mathbb{T}_2^0, \ldots, \mathbb{T}_q^p\}$ is to construct a *hierarchical diagram* or a special tree as follows.



When we learnt first about the tensor symbolized by "$\otimes$" as well as its multilinearity we were left with the problem of developing an intuitive understanding of $\mathbf{x}^1 \otimes \mathbf{x}^2$, $\mathbf{x}^1 \otimes \mathbf{x}_1$ and higher order tensor products. Perhaps it is helpful to represent "the involved vectors" in a contravariant or in a covariant basis. For instances, $\mathbf{x}^1 = \mathbf{e}^1 x_1 + \mathbf{e}^2 x_2 + \mathbf{e}^3 x_3$ or $\mathbf{x}_1 = \mathbf{e}_1 x^1 + \mathbf{e}_2 x^2 + \mathbf{e}_3 x^3$ is a *left* representation of a three-dimensional vector in a *3-left basis* $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}_l$ of contravariant type *or* in a *3-left basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}_l$ of covariant type. Think in terms of $\mathbf{x}^1$ or $\mathbf{x}_1$ as a three-dimensional position vector with right coordinates $\{x_1, x_2, x_3\}$ or $\{x^1, x^2, x^3\}$, respectively. Since the intuitive algebras of vectors is *commutative* we may also represent the three-dimensional vector in a in a *3-right basis* $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}_r$ of contravariant type *or* in a *3-right-basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}_r$ of covariant type such that $\mathbf{x}^1 = x_1\mathbf{e}^1 + x_2\mathbf{e}^2 + x_3\mathbf{e}^3$ or $\mathbf{x}_1 = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3$ is a *right* representation of a three-dimensional position vector which coincides with its *left* representation thanks to *commutability*. Further on, the tensor product $\mathbf{x} \otimes \mathbf{y}$ enjoys the left and right representations

$$(\mathbf{e}^1 x_1 + \mathbf{e}^2 x_2 + \mathbf{e}^3 x_3) \otimes (\mathbf{e}^1 y_1 + \mathbf{e}^2 y_2 + \mathbf{e}^3 y_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{e}^i \otimes \mathbf{e}^j x_i y_i$$
$$\textit{and}$$
$$(x_1\mathbf{e}^1 + x_2\mathbf{e}^2 + x_3\mathbf{e}^3) \otimes (y_1\mathbf{e}^1 + y_2\mathbf{e}^2 + y_3\mathbf{e}^3) = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j \mathbf{e}^i \otimes \mathbf{e}^j$$

which coincides again since we assumed a *commutative algebra* of vectors. The product of coordinates $(x_i y_j), i, j \in \{1,2,3\}$ is often called the *dyadic product*. Please do not miss the alternative covariant representation of the tensor product $\mathbf{x} \otimes \mathbf{y}$ which we introduced so far in the contravariant basis, namely

$$(\mathbf{e}_1 x^1 + \mathbf{e}_2 x^2 + \mathbf{e}_3 x^3) \otimes (\mathbf{e}_1 y^1 + \mathbf{e}_2 y^2 + \mathbf{e}_3 y^3) = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{e}_i \otimes \mathbf{e}_j x^i y^j$$

of left type and

$$(x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3) \otimes (y^1\mathbf{e}_1 + y^2\mathbf{e}_2 + y^3\mathbf{e}_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} x^i y^j \mathbf{e}_i \otimes \mathbf{e}_j$$

of right type. In a similar way we produce

$$\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} = \sum_{i,j,k=1}^{3,3,3} \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k x_i y_j z_k = \sum_{i,j,k=1}^{3,3,3} x_i y_j z_k \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k$$

of contravariant type and

$$\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} = \sum_{i,j,k=1}^{3,3,3} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k x^i y^j z^k = \sum_{i,j,k=1}^{3,3,3} x^i y^j z^k \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

of covariant type. "*Mixed covariant-contravariant*" representations of the tensor product $x_1 \otimes y^1$ are

$$\mathbf{x}_1 \otimes \mathbf{y}^1 = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{e}_i \otimes \mathbf{e}^j \mathbf{x}^i \mathbf{y}_j = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{x}^i \mathbf{y}_j \mathbf{e}_i \otimes \mathbf{e}^j$$
$$\textit{or}$$
$$\mathbf{x}^1 \otimes \mathbf{y}_1 = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{e}^i \otimes \mathbf{e}_j \mathbf{x}_i \mathbf{y}^j = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{x}_i \mathbf{y}^j \mathbf{e}^i \otimes \mathbf{e}_j .$$

In addition, we have to explain the notion $\mathbf{x}^1 \in \mathbb{X}^*, \mathbf{x}_1 \in \mathbb{X}$: While the vector $\mathbf{x}_1$ is an element of the vector space $\mathbb{X}$, $\mathbf{x}^1$ is an element of its dual space. What is a dual space? Indeed the dual space $\mathbb{X}^*$ is the space of *linear functions* over the elements of $\mathbb{X}$. For instance, if the vector space $\mathscr{X}$ is equipped with *inner product*, namely $< \mathbf{e}_i \mid \mathbf{e}_j >= g_{ij}, i, j \in \{1,2,3\}$, with respect to the base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which span the vector space $\mathbb{X}$, then

$$\mathbf{e}^j = \sum_{i=1}^{3} \mathbf{e}_i g^{ij}$$

transforms the covariant base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ into the contravariant base vectors $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}, \mathbb{X}^* = \text{span}\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$, by means of $[g^{ij}] = G^{-1}$, the inverse of the matrix $[g_{ij}] = G \in \mathbb{R}^{3\times 3}$. Similarly the coordinates g_{ij} of the metric tensor g are used for "*raising*" or "*lowering*" the indices of the coordinates x^i, x_j, respectively, for instance

$$x^i = \sum_{j=1}^{3} g^{ij} x_j, x_i = \sum_{j=1}^{3} g_{ij} x^j.$$

In a finite dimensional vector space, the power of a linear space $\mathbb{X}$ and its dual $\mathbb{X}^*$ does not show up. In contrast, in an infinite dimensional vector space $\mathbb{X}$ the dual space $\mathbb{X}^*$ is the space of *linear functions* which play an important role in *functional analysis*. While through the tensor product "$\otimes$" which operated on vectors, e.g. $\mathbf{x} \otimes \mathbf{y}$, we constructed the p-contravariant, q-covariant tensor space or space of multilinear functions $\mathbb{T}_q^p(\mathbb{X}, \mathbb{X}^*)$, e.g. $\mathbb{T}_0^2$, $\mathbb{T}_2^0$, $\mathbb{T}_1^1$, we shall generalize the representation of the elements of $\mathbb{T}_q^p$ by means of

$$f = \sum_{i_1,\dots,i_p=1}^{n=\dim \mathscr{X}^*} \mathbf{e}^{i_1} \otimes \dots \otimes \mathbf{e}^{i_p} f_{i_1 \dots i_p} \in \mathbb{T}_0^p$$

$$f = \sum_{i_1,\dots,i_q=1}^{n=\dim \mathscr{X}^*} \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_q} f^{i_1 \dots i_q} \in \mathbb{T}_q^0$$

$$f = \sum_{i_1,\dots,i_p=1}^{n=\dim \mathscr{X}^*} \sum_{i_1,\dots,i_q=1}^{n=\dim \mathscr{X}} \mathbf{e}^{i_1} \otimes \dots \otimes \mathbf{e}^{i_p} \otimes \mathbf{e}_{j_1} \otimes \dots \otimes \mathbf{e}_{j_q} f_{j_1,\dots,j_q}^{i_1,\dots,i_p} \in \mathbb{T}_q^p$$

for instance

$$f = \sum_{i,j=1}^{3} \mathbf{e}^i \otimes \mathbf{e}^j f_{ij} = \sum_{i,j=1}^{3} f_{ij} \mathbf{e}^i \otimes \mathbf{e}^j \in \mathbb{T}_0^2$$

$$f = \sum_{i,j=1}^{3} \mathbf{e}_i \otimes \mathbf{e}_j f^{ij} = \sum_{i,j=1}^{3} f^{ij} \mathbf{e}_i \otimes \mathbf{e}_j \in \mathbb{T}_2^0$$

$$f = \sum_{i,j=1}^{3} \mathbf{e}^i \otimes \mathbf{e}_j f_j^i = \sum_{i,j=1}^{3} f_j^i \mathbf{e}^i \otimes \mathbf{e}_j \in \mathbb{T}_1^1$$

We have to emphasize that the tensor coordinates $f_{i_1,\ldots,i_p}$, $f^{i_1,\ldots,i_q}$, $f^{i_1,\ldots,i_p}_{j_1,\ldots,j_p}$ are no longer of dyadic or product type. For instance, for

(2,0)-tensor: trilinear functions:

$$f = \sum_{i,j=1}^{n} \mathbf{e}^i \otimes \mathbf{e}^j f_{ij} = \sum_{i,j=1}^{n} f_{ij} \mathbf{e}^i \otimes \mathbf{e}^j \in \mathbb{T}_0^2$$

$$f_{ij} \neq f_i f_j$$

(2,1)-tensor: trilinear functions:

$$f = \sum_{i,j,k=1}^{n} \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}_k f_{ij}^k = \sum_{i,j,k=1}^{n} f_{ij}^k \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}_k \in \mathbb{T}_1^2$$

$$f_{ij}^k \neq f_i f_j f^k$$

(3,1)-tensor: ternary functions:

$$f = \sum_{i,j,k,l=1}^{n} \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}_l f_{ijk}^l = \sum_{i,j,k,l=1}^{n} f_{ijk}^l \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}_l \in \mathbb{T}_1^3$$

$$f_{ijk}^l \neq f_i f_j f_k f^l$$

holds. *Table 1-1* is a list of (p,q)-tensors as they appear in various sciences. Of special importance is the decomposition of multilinear functions as elements of the space $\mathbb{T}_q^p$ into their symmetric, antisymmetric and residual constituents we are going to outline.

Table 1-1: Various examples of tensor spaces $\mathbb{T}_q^p$ ($p+q$: rank of tensor) (2,0) tensor, tensor space $\mathbb{T}_0^2$

Metric tensor Gauss curvature tensor Ricci curvature tensor	differential geometry
gravity gradient tensor	gravitation
Faraday tensor, Maxwell tensor tensor of dielectric constant tensor of permeability	electromagnetic
strain tensor, stress tensor	continuum mechanics
energy momentum tensor	mechanics electromagnetism electrostatics
2nd order multipole tensor	gravitostatics magnetostatics electrostatics
variance-covariance matrix	mathematical statistics

Table 1-2: Various examples of tensor spaces $\mathbb{T}_q^p$ ($p+q$: rank of tensor) (2,1) tensor, tensor space $\mathbb{T}_1^2$

Cartan torsion tensor	differential geometry
3^{rd} order multipole tensor	gravitostatics magnetostatics electrostatics
skewness tensor 3^{rd} momentum tensor of probability distribution	mathematical statistics
tensor of piezoelectric constant	coupling of stress and electrostatic field

Table 1-3: Various examples of tensor spaces $\mathbb{T}_q^p$ (p+q : rank of tensor) (3,1)tensor, (2,2) tensor, tensor space $\mathbb{T}_1^3$, $\mathbb{T}_2^2$

Reimann curvature tensor	differential geometry
4^{th} order multipole tensor	gravitostatics magnetostatics electrostatics
Hooke tensor	stress-strain relation constitutive equation continuum mechanics elasticity, viscosity
kurtosis tensor 4^{th} moment tensor of a probability distribution	mathematical statistics

Scholia

A beautiful introduction into *multilinear superalgebra* based upon left and right super modules with *left and right tensor coordinates* –not coinciding- is given by *F. Constantinescu and H. F. de Groote* (1984). Applications in "*supersymmetric physics*" are highlighted: Supersymmetry is a symmetry between *bosons*, elementary particles with integer spin, and *fermions*, elementary particles with half-integral spin.

1-2 Decomposition of multilinear functions into symmetric multilinear functions, antisymmetric multi-linear functions and residual multilinear functions: $\mathbb{T}_q^p = \mathbb{S}_q^p \otimes \mathbb{A}_q^p \otimes \mathbb{R}_q^p$

$\mathbb{T}_q^p$ as the space of multilinear functions follows the decomposition $\mathbb{T}_q^p = \mathbb{S}_q^p \otimes \mathbb{A}_q^p \otimes \mathbb{R}_q^p$ into the subspace $\mathbb{S}_q^p$ of *symmetric* multilinear functions, the subspace $\mathbb{A}_q^p$ of *antisymmetric* multilinear functions and the subspace $\mathbb{R}_q^p$ of *residual* multilinear functions:

Box 1.2i: Antisymmetry of the symbols $f_{i_1 \ldots i_p}$

$$f_{ij} = -f_{ji}$$

$$f_{ijk} = -f_{ikj},\ f_{jki} = -f_{jik},\ f_{kij} = -f_{kji}$$

$$f_{ijkl} = -f_{ijlk},\ f_{jkli} = -f_{jkil},\ f_{klij} = -f_{klji},\ f_{kijk} = -f_{likj}$$

Box 1.2ii: Symmetry of the symbols $f_{i_1 \ldots i_p}$

$$f_{ij} = f_{ji}$$

$$f_{ijk} = f_{ikj},\ f_{jki} = f_{jik},\ f_{kij} = f_{kji}$$

$$f_{ijkl} = f_{ijlk},\ f_{jkli} = f_{jkil},\ f_{klij} = f_{klji},\ f_{kijk} = f_{likj}$$

Box 1.2iii: The interior product of bases of $\mathbb{S}^p, n = \dim \mathbb{X} = \dim \mathbb{X}^* = 3$

$$\mathbb{S}^1 : \frac{1}{1!}\mathbf{e}^i$$

$$\mathbb{S}^2 : \frac{1}{2!}(\mathbf{e}^i \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^i) =: \mathbf{e}^i \vee \mathbf{e}^j,\ \ \mathbf{e}^i \vee \mathbf{e}^j = +\mathbf{e}^j \vee \mathbf{e}^i$$

$$\begin{aligned}\mathbb{S}^3 : \frac{1}{3!}(&\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k + \mathbf{e}^i \otimes \mathbf{e}^k \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}^i + \\ &+ \mathbf{e}^j \otimes \mathbf{e}^i \otimes \mathbf{e}^k + \mathbf{e}^k \otimes \mathbf{e}^i \otimes \mathbf{e}^j + \mathbf{e}^k \otimes \mathbf{e}^j \otimes \mathbf{e}^i) := \\ &=: \mathbf{e}^i \vee \mathbf{e}^j \vee \mathbf{e}^k\end{aligned}$$

$$\mathbf{e}^i \vee \mathbf{e}^j \vee \mathbf{e}^k = \mathbf{e}^i \vee \mathbf{e}^k \vee \mathbf{e}^j = \mathbf{e}^k \vee \mathbf{e}^i \vee \mathbf{e}^j = \mathbf{e}^k \vee \mathbf{e}^j \vee \mathbf{e}^i = \mathbf{e}^j \vee \mathbf{e}^k \vee \mathbf{e}^i = \mathbf{e}^j \vee \mathbf{e}^i \vee \mathbf{e}^k$$

Box 1.2iv: The exterior product of bases of $\mathbb{A}^p, n = \dim \mathbb{X} = \dim \mathbb{X}^* = 3$

$$\mathbb{A}^1 : \frac{1}{1!}\mathbf{e}^i$$

$$\mathbb{A}^2 : \frac{1}{2!}\left(\mathbf{e}^i \otimes \mathbf{e}^j - \mathbf{e}^j \otimes \mathbf{e}^i\right) =: \mathbf{e}^i \wedge \mathbf{e}^j,\ \mathbf{e}^i \wedge \mathbf{e}^j = -\mathbf{e}^j \wedge \mathbf{e}^i$$

$$\begin{aligned}\mathbb{A}^3 : \frac{1}{3!}\Big(&\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k - \mathbf{e}^i \otimes \mathbf{e}^k \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}^i - \\ &- \mathbf{e}^j \otimes \mathbf{e}^i \otimes \mathbf{e}^k + \mathbf{e}^k \otimes \mathbf{e}^i \otimes \mathbf{e}^j - \mathbf{e}^k \otimes \mathbf{e}^j \otimes \mathbf{e}^i\Big) = \\ &=: \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k\end{aligned}$$

$$\begin{aligned}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k &= -\mathbf{e}^i \wedge \mathbf{e}^k \wedge \mathbf{e}^j = +\mathbf{e}^k \wedge \mathbf{e}^i \wedge \mathbf{e}^j = -\mathbf{e}^k \wedge \mathbf{e}^j \wedge \mathbf{e}^i = \\ &= +\mathbf{e}^j \wedge \mathbf{e}^k \wedge \mathbf{e}^i = -\mathbf{e}^j \wedge \mathbf{e}^i \wedge \mathbf{e}^k\end{aligned}$$

Box 1.2v: $\mathbb{S}^p$, symmetric multilinear functions

$$\mathbb{T}_0^1 \supset \mathbb{S}^1 \ni f = \{\sum_{i=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^i f_i\}$$

$$\mathbb{T}_0^2 \supset \mathbb{S}^2 \ni f = \{\frac{1}{2!}\sum_{i,j=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \vee \mathbf{e}^j f_{ij}\} = \{\sum_{i \le j}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \vee \mathbf{e}^j f_{(ij)} \mid f_{(ij)} := \frac{1}{2!}(f_{(ij)} + f_{(ji)})\}$$

$$\mathbb{T}_0^3 \supset \mathbb{S}^3 \ni f = \{\frac{1}{3!}\sum_{i,j,k=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \vee \mathbf{e}^j \vee \mathbf{e}^k f_{ijk}\} = \{\sum_{i<j<\kappa}^{n} \mathbf{e}^i \vee \mathbf{e}^j \vee \mathbf{e}^k f_{(ijk)} \mid f_{(ijk)} :=$$

$$:= \frac{1}{3!}(f_{ijk} + f_{ikj} + f_{jki} + f_{jik} + f_{kij} + f_{kji})\}$$

$$\mathbb{T}_0^p \supset \mathbb{S}^p \ni f = \{\frac{1}{p!}\sum_{i_1,i_2,\ldots,i_p=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \vee \mathbf{e}^{i_2} \vee \ldots \vee \mathbf{e}^{i_p} f_{i_1 i_2 \ldots i_p}\} =$$

$$= \{\sum_{i_1 \le i_2 \le \ldots \le i_p}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \vee \mathbf{e}^{i_2} \vee \ldots \vee \mathbf{e}^{i_p} f_{(i_1 i_2 \ldots i_p)} \mid$$

$$\mid f_{(i_1 i_2 \ldots i_p)} := \frac{1}{p!}(f_{i_1 \ldots i_{p-1} i_p} + f_{i_1 \ldots i_p i_{p-1},} + \ldots + f_{i_p \ldots i_1 i_2} + f_{i_p \ldots i_2 i_1})\}$$

Lemma 1-1:
$\dim \mathbb{S}^p = \binom{n+p+1}{p}$, in particular if $n = p$, then $\dim \mathbb{S}^p = \binom{2p-1}{p}$

Box 1.2vi: $\mathbb{A}^p$, antisymmetric multilinear functions

$$\mathbb{T}_0^1 \supset \mathbb{A}^1 \ni f = \{\sum_{i=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^i f_i\}$$

$$\mathbb{T}_0^2 \supset \mathbb{A}^2 \ni f = \{\frac{1}{2!}\sum_{i,j=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij}\} =$$

$$= \{\sum_{i \le j}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j f_{(ij)} \mid f_{(ij)} := \frac{1}{2!}(f_{(ij)} + f_{(ji)})\}$$

$$\mathbb{T}_0^3 \supset \mathbb{A} \ni f = \{\frac{1}{3!}\sum_{i,j,k=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k f_{ijk}\} =$$

$$= \{\sum_{i<j<\kappa}^{n} \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k f_{(ijk)} \mid f_{(ijk)} :=$$

$$:= \frac{1}{3!}(f_{ijk} - f_{ikj} + f_{jki} - f_{jik} + f_{kij} - f_{kji})\}$$

$$\mathbb{T}_0^p \supset \mathbb{A}^p \ni f = \{\frac{1}{p!}\sum_{i_1,i_2,\ldots,i_p=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \ldots \wedge \mathbf{e}^{i_p} f_{i_1 i_2 \ldots i_p}\} =$$

$$= \{\sum_{i_1 < i_2 < \ldots < i_p}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \ldots \wedge \mathbf{e}^{i_p} f \mid f_{(i_1 i_2 \ldots i_p)} :=$$

$$:= \frac{1}{p!}(f_{i_1 \ldots i_{p-1} i_p} - f_{i_1 \ldots i_p i_{p-1},} + \ldots + f_{i_p \ldots i_1 i_2} - f_{i_p \ldots i_2 i_1})\}.$$

Lemma 1-2:
$\dim \mathbb{A}^p = n!/(p!(n-p)!) = \binom{n}{p}$, in particular if $n = p$, then $\dim \mathbb{A}^p = 1$.

Box.1.2vii: $\mathbb{A}_q$ antisymmetric multilinear functions, exterior product

(i) For every $\mathbf{x}_1,...,\mathbf{x}_{i-1},...,\mathbf{x}_q \in \mathbb{X}$ as well as and $r,s \in \mathbb{R}$ multilinearity implies

$$\mathbf{x}_1 \wedge ... \wedge \mathbf{x}_{i-1} \wedge (r\mathbf{x}+s\mathbf{y}) \wedge \mathbf{x}_{i+1} \wedge ... \wedge \mathbf{x}_q = \\ = r(\mathbf{x}_1 \wedge ... \wedge \mathbf{x}_{i-1} \wedge \mathbf{x} \wedge \mathbf{x}_{i+1} \wedge ... \wedge \mathbf{x}_q) + \\ + s(\mathbf{x}_1 \wedge ... \wedge \mathbf{x}_{i-1} \wedge \mathbf{y} \wedge \mathbf{x}_{i+1} \wedge ... \wedge \mathbf{x}_q).$$

(ii) For every permutation σ of $\{1,2,...,q\}$ we have

$$\mathbf{x}_{\sigma_1} \wedge \mathbf{x}_{\sigma_2} ... \wedge \mathbf{x}_{\sigma q} = sign(\sigma)\, \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge ... \wedge \mathbf{x}_q$$

(iii) Let $\mathbb{A} \in \mathbb{A}_q(\mathbb{X}), B \in \mathbb{A}_s(\mathbb{X})$; then

$$B \wedge \mathbb{A} = (-1)^{qs}\, \mathbb{A} \wedge B$$

(iv) For every $q, 0 \le q \le n$, the tensor space $\mathbb{A}_q$ of antisymmetric multilinear functions has dimension

$$\dim \mathbb{A}_q = \binom{n}{q} = n!/(q!(n-q)!).$$

As detailed examples we like to decompose $\mathbb{T}_0^1$, $\mathbb{T}_1^0, \mathbb{T}_0^2$, $\mathbb{T}_2^0$ in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, into symmetric and antisymmetric constituents.

Example 1-2: $\mathbb{T}_q^p = \mathbb{S}_q^p \otimes \mathbb{A}_q^p \otimes \mathbb{R}_q^p$ decomposition of multilinear functions into symmetric and antisymmetric constituents.

As a *first example* of the decomposition of multilinear functions (tensor space) into *symmetric* and *antisymmetric* constituents we consider a linear space $\mathbb{X}$ (vector space) of dimension $\dim \mathbb{X} = n = 2$. Its dual space $\mathbb{X}^*$, $\dim \mathbb{X}^* = \dim \mathbb{X} = n = 2$, is spanned by orthonormal contravariant base vectors $\{e^1, e^2\}$. Choose q=0, p=1 and p=2.

$$\mathbb{X} = span\{\mathbf{e}^1, \mathbf{e}^2\} \qquad versus \qquad \mathbb{X}^* = span\{\mathbf{e}^1, \mathbf{e}^2\}$$

$$\mathbb{T}_0^1 = \mathbb{A}^1 = \mathbb{S}^1 \in f = \{\sum_{i=1}^{2} \mathbf{e}^i f_i\} = \mathbf{e}^1 f_1 + \mathbf{e}^2 f_2 \in \mathbb{X}$$

$$\mathbb{T}_0^2 = \mathbb{A}^2 \oplus \mathbb{S}^2$$

$$\mathbb{T}_0^2 \ni f = \{\sum_{i,j=1}^{2} \mathbf{e}^i \otimes \mathbf{e}^j f_{ij}\} = \mathbf{e}^1 \otimes \mathbf{e}^1 f_{11} + \mathbf{e}^1 \otimes \mathbf{e}^2 f_{12} + \mathbf{e}^2 \otimes \mathbf{e}^1 f_{21} + \mathbf{e}^2 \otimes \mathbf{e}^2 f_{22} =$$

$$= \mathbf{e}^1 \otimes \mathbf{e}^1 f_{11} + \mathbf{e}^2 \otimes \mathbf{e}^2 f_{22} + \frac{1}{2}(\mathbf{e}^1 \otimes \mathbf{e}^2 - \mathbf{e}^2 \otimes \mathbf{e}^1) f_{12} + \frac{1}{2}(\mathbf{e}^1 \otimes \mathbf{e}^2 + \mathbf{e}^2 \otimes \mathbf{e}^1) f_{12} -$$

$$- \frac{1}{2}(\mathbf{e}^1 \otimes \mathbf{e}^2 - \mathbf{e}^2 \otimes \mathbf{e}^1) f_{21} + \frac{1}{2}(\mathbf{e}^1 \otimes \mathbf{e}^2 + \mathbf{e}^2 \otimes \mathbf{e}^1) f_{21} =$$

$$= \mathbf{e}^1 \vee \mathbf{e}^1 f_{11} + \mathbf{e}^2 \vee \mathbf{e}^2 f_{22} + \mathbf{e}^1 \wedge \mathbf{e}^2 (f_{12} - f_{21})/2 + \mathbf{e}^1 \wedge \mathbf{e}^2 (f_{12} + f_{21})/2$$

$$\dim \mathbb{S}^2 = 3, \qquad \dim \mathbb{A}^2 = 1.$$

As a *second example* of the decomposition of multilinear functions (tensor space) into symmetric and antisymmetric constituents we consider a linear space $\mathbb{X}$ (vector space) of dimension $\dim \mathbb{X} = n = 3$, spanned by orthonormal contravariant base vectors $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$. Choose p = 0, q = 1 and 2.

$$\mathbb{X} = \operatorname{span}\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$$

$$\mathbb{T}_1^0 = \mathbb{A}_1 = \mathbb{S}_1 \ni f = \{\sum_{i=1}^{3} \mathbf{e}_1 f^i\} = \mathbf{e}_1 f^1 + \mathbf{e}_1 f^2 + \mathbf{e}_1 f^3 \in \mathbb{X} \tag{0.1}$$

$$\mathbb{T}_2^0 = \mathbb{A}_2 \oplus \mathbb{S}_2$$

$$\mathbb{T}_2^0 \ni f = \{\sum_{i,j=1}^{3} \mathbf{e}_i \otimes \mathbf{e}_j f^{ij}\} = \{\sum_{i=1}^{3} \mathbf{e}_i \otimes \mathbf{e}_1 f^{i1} + \sum_{i=1}^{3} \mathbf{e}_i \otimes \mathbf{e}_2 f^{i2} + \sum_{i=1}^{3} \mathbf{e}_i \otimes \mathbf{e}_3 f^{i3}\} =$$

$$= \mathbf{e}_1 \otimes \mathbf{e}_1 f^{11} + \mathbf{e}_2 \otimes \mathbf{e}_1 f^{21} + \mathbf{e}_3 \otimes \mathbf{e}_1 f^{31} + \mathbf{e}_1 \otimes \mathbf{e}_2 f^{12} + \mathbf{e}_2 \otimes \mathbf{e}_2 f^{22} +$$

$$+ \mathbf{e}_3 \otimes \mathbf{e}_2 f^{32} + \mathbf{e}_1 \otimes \mathbf{e}_3 f^{13} + \mathbf{e}_2 \otimes \mathbf{e}_3 f^{23} + \mathbf{e}_3 \otimes \mathbf{e}_3 f^{33} =$$

$$+\frac{1}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) f^{12} + \frac{1}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) f^{12} - \frac{1}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) f^{21} +$$

$$+\frac{1}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) f^{12} + \frac{1}{2}(\mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2) f^{23} + \frac{1}{2}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) f^{23} -$$

$$-\frac{1}{2}(\mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2) f^{32} + \frac{1}{2}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) f^{32} + \frac{1}{2}(\mathbf{e}_3 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_3) f^{31} +$$

$$+\frac{1}{2}(\mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3) f^{31} - \frac{1}{2}(\mathbf{e}_3 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_3) f^{13} + \frac{1}{2}(\mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3) f^{31}.$$

Since the subspaces $\mathbb{S}_q^p, \mathbb{A}_q^p$ and $\mathbb{R}_q^p$ are independents, $\mathbb{S}_q^p \oplus \mathbb{A}_q^p \oplus \mathbb{R}_q^p$ denotes the *direct sum* of subspace $\mathbb{S}_q^p, \mathbb{A}_q^p$ and $\mathbb{R}_q^p$. Unfortunately $\mathbb{T}_q^p$ as the space of multilinear functions cannot be completely decomposed in the space of symmetric multilinear functions: for instance, the dimension identities apply $\dim \mathbb{T}^p = n^p$, $\dim \mathbb{S}^p = \binom{n+p-1}{p}$, $\dim \mathbb{A}^p = \binom{n}{p}$ with respect of a vector space $\mathbb{X}$ of dimension $\dim \mathscr{X} = n$, such that $\dim \mathbb{R}^p = \dim \mathbb{T}^p - \dim \mathbb{S}^p - \dim \mathbb{A}^p =$ $= n^p - \binom{n+p+1}{p} - \binom{n}{p} < n^p$, in general. There is *one exception,* namely the (2,0) or (1,1) or (0,2) tensor space where the dimension of the subspace $\mathbb{R}_0^2, \mathbb{R}_1^1$, $\mathbb{R}_2^0$ of *residual* multilinear functions *is zero*. An example is $\dim \mathbb{R}^2 =$ $= n^2 - \binom{n+1}{p} - \binom{n}{2} = n^2 - (n+1)n/2 - n(n-1)^2 = 0$.

1-3 Matrix algebra, array algebra, matrix norm and inner product

Symmetry and antisymmetry of the symbols $f_{i_1 \dots i_p}$ can be visualized by the trees of *Box 1.2i* and *Box 1.2ii*. With respect to the symbols of the *interior product* "V" and the *exterior* " Λ " ("*wedge product*") we are able to redefine symmetric antisymmetric functions according to *Box1.2iii-vi*. Note the *isomorphism of tensor algebra* $\mathbb{T}_q^p$ and *array algebra*, namely of

(i)	$[f_i] \in \mathbb{R}^n$	(one-dimensional array, "column vector", $\dim[f_i] = n \times 1$)
(ii)	$[f_{ij}] \in \mathbb{R}^{n\times n}$	(two-dimensional array, column-row array, "matrix", $\dim[f_{ij}] = n \times n$)
(iii)	$[f_{ijk}] \in \mathbb{R}^{n\times n\times n}$	(three-dimensional array, "indexed-matrix", $\dim[f_{ijk}] = n \times n \times n$)

etc. For the base space $\mathbf{x} \in \mathbf{\Omega} \subset \mathbb{R}^3$ to be three-dimensional *Euclidian* we had answered the question how to measure the length of a vector („norm") and the angle between two vectors („inner product"). The same question will finally been raised for tensors $t_q^p \in \mathbb{T}_q^p$. The answer is *constructively* based on the *vectorization* of the arrays $[f_{ij}]$, $[f_{ijk}]$, ... $[f_{i_1 \dots i_p}]$ by taking advantage of the symmetry-antisymmetry structure of the arrays and later on applying the *Euclidean* norm and the *Euclidean* inner product to the *vectorized array.*

For a 2-contravariant, 0-covariant tensor we shall outline the procedure.

(i) *Firstly* let $\boldsymbol{F}=[f_{ij}]$ be the quadratic matrix of dimension $\dim F = n \times n$, an element of $\mathbb{T}_0^2$. Accordingly vec **F** is the vector

$$vec\mathbf{F} = \begin{bmatrix} f_{i_1} \\ f_{i_2} \\ \vdots \\ f_{i_{n-1}} \\ f_{i_n} \end{bmatrix}, \ \dim \operatorname{vec} \mathbf{F} = n^2 \times 1$$

which is generated by stacking the elements of the matrix **F** *columnwise* in a vector. The *Euclidean* norm and the *Euclidean* inner product of vec**F**, vec**G,** respectively is

$$||\operatorname{vec}\mathbf{F}||^2 := (\operatorname{vec}\mathbf{F})^T(\operatorname{vec}\mathbf{F}) = \operatorname{tr}\mathbf{F}^T\mathbf{F},$$

$$<\operatorname{vec}\mathbf{F}|\operatorname{vec}\mathbf{G}> := (\operatorname{vec}\mathbf{F})^T \operatorname{vec}\mathbf{G} = \operatorname{tr}\mathbf{F}^T\mathbf{G}.$$

(ii) *Secondly* let $\mathbf{F}= [f_{ij}]= [f_{ji}]$ be the *symmetric* matrix of dimension $\dim\mathbf{F}= n \times n$, an element of $\mathbb{S}^2$. Accordingly vech**F** (read „vector half") is the n(n+1)/2x1 vector which is generated by stacking the elements *on an under* the main diagonal of the matrix **F** *columnwise* in a vector:

$$\mathbf{F}=\left[f_{ij}\right]=\left[f_{ji}\right]=\mathbf{F}^T \Rightarrow \text{vech}\,\mathbf{F}:=\begin{bmatrix} f_{11} \\ \vdots \\ \underline{f_{1n}} \\ f_{22} \\ \vdots \\ \underline{f_{2n}} \\ \vdots \\ f_{nn} \end{bmatrix}, \dim \text{vech}\,\mathbf{F}=n(n+1)/2$$

$$\text{vech}\mathbf{F}=\mathbf{H}\text{vech}\ \mathbf{F},\ \dim\mathbf{H}=n(n+1)/2\times n^2.$$

The *Euclidean* norm and the *Euclidean* inner product of vech**F**, vech**G**, respectively is

$$\left.\begin{array}{l}\mathbf{F}=[f_{ij}]=[f_{ji}]=\mathbf{F}^T \\ \mathbf{G}=[g_{ij}]=[g_{ji}]=\mathbf{G}^T\end{array}\right]\Rightarrow$$

$$\|\text{vech}\ \mathbf{F}\|^2:=(\text{vech}\ \mathbf{F})^T(\text{vech}\ \mathbf{F})$$
$$<\text{vech}\ \mathbf{F}|\text{vech}\ \mathbf{G}>:=(\text{vech}\ \mathbf{F})^T\text{vech}\ \mathbf{G}$$

(iii) Thirdly let **F**= $[f_{ij}]$= - $[f_{ji}]$ be the antisymmetric matrix of dimension dim**F** = $n\times n$, an element of $\mathbb{A}^2$. Accordingly veck**F** (*read* “vector skew“) is the *n(n-1)/2x 1* vector which is generated by stacking the elements *under* the main diagonal of the matrix **F** columnwise in a vector:

$$\mathbf{F}=[f_{ij}]=-[f_{ji}]=-\mathbf{F}^T \Rightarrow \text{veck}\,\mathbf{F}:=\begin{bmatrix} f_{21} \\ \vdots \\ \underline{f_{n1}} \\ f_{32} \\ \vdots \\ \underline{f_{n2}} \\ \vdots \\ f_{n-1n} \end{bmatrix}, \dim \text{veck}\,\mathbf{F}=n(n-1)/2$$

$$\text{veck}\ \mathbf{F}=\mathbf{K}\ \text{vec}\ \mathbf{F},\ \dim\ \mathbf{K}=n(n+1)/2\times n^2.$$

The *Euclidean* norm and the *Euclidean* inner product of veck **F**, veck **G**, respectively is

$$\left.\begin{array}{l}\mathbf{F}=\left[f_{ij}\right]=-\left[f_{ji}\right]=-\mathbf{F}^T \\ \mathbf{G}=\left[g_{ij}\right]=-\left[g_{ji}\right]=-\mathbf{G}^T\end{array}\right]\Rightarrow$$

$$\|\text{veck}\ \mathbf{F}\|^2:=(\text{veck}\ \mathbf{F})^T(\text{veck}\ \mathbf{F})$$
$$<\text{veck}\ \mathbf{F}|\text{veck}\ \mathbf{G}>:=(\text{veck}\ \mathbf{F})^T\text{veck}\ \mathbf{G}$$

Example 1-3: Norm and inner product of a 2-contravariant, 0-covariant tensor

(i)
$$\mathbf{A} := \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix}, \dim\mathbf{A} = 3\text{x}3 \Rightarrow$$

$$\text{vec}\mathbf{A} = [a, b, c, d, e, f, g, h, k]^T, \dim \text{vec}\mathbf{A} = 9\times 1$$
$$\|\text{vec}\mathbf{A}\|^2 = (\text{vec}\mathbf{A})^T(\text{vec}\mathbf{A}) = \text{tr}\,\mathbf{A}^T\mathbf{A} = a^2 + \ldots + k^2$$

(ii)
$$\mathbf{A} := \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} = \mathbf{A}^T, \dim\mathbf{A} = 3\times 3 \Rightarrow$$

$$\text{vech}\mathbf{A} = [a, b, c, d, e, f]^T, \dim \text{vech}\mathbf{A} = 6\text{x}1$$

$$\text{vech}\mathbf{A} = \mathbf{H}\text{vech}\mathbf{A}$$

$$\forall\mathbf{H} := \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \dim\mathbf{H} = 6\text{x}9$$

$$\|\text{vech}\mathbf{A}\|^2 = (\text{vech}\mathbf{A})^T(\text{vech}\mathbf{A}) = a^2 + b^2 + c^2 + d^2 + e^2 + f^2$$

(H.V. Henderson and S.A. Searle, 1978, p.68-69)

(iii)
$$\mathbf{A} := \begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{bmatrix} = -\mathbf{A}^T, \dim\mathbf{A} = 3\times 3 \Rightarrow$$

$$\text{veck}\mathbf{A} = [a, b, c, d, e, f]^T, \dim \text{veck}\mathbf{A} = 6\times 1$$

$$\text{veck}\mathbf{A} = \mathbf{K}\text{veck}\mathbf{A}$$

$$\forall\mathbf{K} := \frac{1}{2}\left[\begin{array}{cccc|cccc|cccc|cccc} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right],$$

$$\dim\mathbf{K} = 6\times 16$$

$$\|\text{veck}\mathbf{A}\|^2 = a^2 + b^2 + c^2 + d^2 + e^2 + f^2$$ ♣

1-4 The Hodge star operator, self duality

The most important operator of the algebra of antisymmetric multilinear functions is the „*Hodge star operator*“ which we shall present finally. In addition, we shall bring to you the surprising special feature of *skew algebra* called „*self-duality*“.

The algebra $\mathbb{A}_q^p$ of antisymmetric multilinear functions has been based on the exterior product “$\wedge$“ (“wedge product“). There has been created a duality operator called the *Hodge star operator* $*$ which is a linear map of $\mathbb{A}^p \to \mathbb{A}^{n-p}$ where $n = \dim \mathbb{X} = \dim \mathbb{X}^*$ denotes the dimension of the *base space* $\mathbb{X} = \dim \mathbb{R}^3$. The basic idea of such a map of antisymmetric multilinear functions $f \in \mathbb{A}^p$ into antisymmetric linear functions $*f \in \mathbb{A}^{n-p}$ originates according to *Box 1.2vii* from the following situation: The multilinear base of $\mathbb{A}^p$ is spanned by

$$\{1, \mathbf{e}^{i_1}, \mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2}, \ldots \mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \ldots \mathbf{e}^{i_p}\}$$

once we focus on *p=0,1,2,....n*, respectively. Obviously for any dimension number n and *p*-contravariant, *q*-covariant index of the skew tensor space $\mathbb{A}_q^p$ there is an associated cobasis, namely

$$n = 1, p = 0,1: \begin{cases} basis: \{1, \mathbf{e}^{i_1}\} \\ associated\ \ cobasis: \{1, \mathbf{e}^{i_1}\} \end{cases}$$

$$n = 2, p = 0,1,2: \begin{cases} basis: \{1, \mathbf{e}^{i_1}, \mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2}\} \\ associated\ \ cobasis: \{\mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2}, \mathbf{e}^{i_2}, 1\} \end{cases}$$

$$n = 3, p = 0,1,2,3: \begin{cases} basis: \{1, \mathbf{e}^{i_1}, \mathbf{e}^{i_2} \wedge \mathbf{e}^{i_3}, \mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \mathbf{e}^{i_3}\} \\ associated\ \ cobasis: \{\mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \mathbf{e}^{i_3}, \mathbf{e}^{i_2} \wedge \mathbf{e}^{i_3}, \mathbf{e}^{i_3}, 1\} \end{cases}$$

in general , for arbitrary $n \in \mathbb{N}$, *p=0,1,...,n-1,n*

Basis:

$$\{1, \mathbf{e}^{i_1}, \ldots, \mathbf{e}^{i_1} \wedge \ldots \wedge \mathbf{e}^{i_n}\}$$

Associated cobasis:

$$\{\mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \ldots \wedge \mathbf{e}^{i_{n-1}} \wedge \mathbf{e}^{i_n}, \mathbf{e}^{i_2} \wedge \ldots \wedge \mathbf{e}^{i_{n-1}} \wedge \mathbf{e}^{i_n}, \ldots, \mathbf{e}^{i_{n-1}} \wedge \mathbf{e}^{i_n}, \mathbf{e}^{i_n}, 1\}$$

as long as we concentrate on *p*-contravariant $\mathbb{A}^p$ only. A similar set-up of basis-associated cobasis for q-covariant $\mathbb{A}_q$ and mixed $\mathbb{A}_q^p$ can be made. The linear map $\mathbb{A}^p \to \mathbb{A}^{n-p}$, the *Hodge star operator*

$$*(\mathbf{e}^{i_1} \wedge \ldots \wedge \mathbf{e}^{i_n}) := \frac{1}{(n-p)!} \mathbf{e}^{i_1 \ldots i_p}_{i_{p+1} \ldots i_n} \mathbf{e}^{i_{p+1}} \wedge \ldots \wedge \mathbf{e}^{i_n}$$

maps by means of the permutation symbol

$$\varepsilon^{i_1 \dots i_p}_{i_{p+1} \dots i_n} := \begin{bmatrix} +1 \ \textit{for an even permutation of } \{1,2,...,n-1,n\} \\ -1 \ \textit{for an odd permutation of } \{1,2,...,n-1,n\} \\ 0 \ \textit{otherwise} \end{bmatrix}$$

-sometimes called *Eddington's epsilons* – on orthonormal („unimodular") base of $\mathbb{A}^p$ onto an orthonormal („unimodular") base of $\mathbb{A}^{n-p}$.

For asymmetric multilinear functions also called antisymmetric tensor-valued functions represented in an orthonormal („unimodular") base the *Hodge star operator* is the following linear map

$$\mathbb{T}^p_0 \supset \mathbb{A}^p \ni f = \{\frac{1}{p!} \sum_{i_1,\dots,i_p=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \wedge \dots \wedge \mathbf{e}^{i_p} f_{i_1 \dots i_p}\}$$

$$^*\mathbb{T}^p_0 \supset \mathbb{A}^{n-p} \ni *f = \{\frac{1}{(n-p)!} \sum_{i_{p+1},\dots,i_n}^{n=\dim \mathbb{X}^*} \sum_{i_1,\dots,i_p}^{n=\dim \mathbb{X}} \frac{1}{p!} \varepsilon^{i_1 \dots i_p}_{i_{p+1} \dots i_n} \mathbf{e}^{i_{p+1}} \wedge \dots \wedge \mathbf{e}^{i_n} f_{i_1 \dots i_p}\}.$$

As soon as the base space $\mathbf{x} \in \mathbf{\Omega} \subset \mathbb{R}^3$ is *not* covered by Cartesian coordinates, rather by *curvilinear coordinates,* its coordinates base

$$\{b^1, b^2, b^3\} = \{dy^1, dy^2, dy^3\} \ \textit{versus} \ \{b^1, b^2, b^3\} = \{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}\}$$

of contravariant *versus* covariant type is covariant type is *neither* orthogonal *nor* normalized. It is for this reason that finally we present **f*, the *Hodge star operator* of an antisymmetric multilinear function *f,* also called *the dual of f,* in a general coordinate base.

Definition 1-2 (**Hodge star operator**, the dual of an antisymmetric multilinear function)

If an antisymmetric (p, 0) multilinear function is an element of the skew algebra $\mathbb{A}^p$ *with respect, to a general base* $\{\mathbf{b}^{i_1} \wedge \dots \wedge \mathbf{b}^{i_p}\}$ *is given*

$$f = \{\frac{1}{p!} \sum_{i_1,\dots,i_p=1}^{n=\dim \mathbb{X}^*} \mathbf{b}^{i_1} \wedge \dots \wedge \mathbf{b}^{i_p} f_{i_1 \dots i_p}\}$$

then the *Hodge star operator*, the *dual* of *f*, can be uniquely represented by

(i)

$$*f = \{\frac{1}{(n-p)!} \sum_{i_{p+1},\dots,i_n}^{n=\dim \mathbb{X}^*} \sum_{i_1,\dots,i_p}^{n=\dim \mathbb{X}^*} \sum_{j_1,\dots,j_p}^{n=\dim \mathbb{X}^*} \frac{1}{p!} \mathbf{b}^{i_{p+1}} \wedge \dots \wedge \mathbf{b}^{i_p} \sqrt{g} \varepsilon_{i_1 \dots i_p i_{p+1} \dots i_n} g^{i_1 j_1} \dots g^{i_p j_p} f_{i_1 \dots i_p}\}$$

(ii) $$*f=\{\frac{1}{(n-p)!}\sum_{i_{p+1},\dots,i_n}^{n=\dim\mathbb{X}^*}\sum_{i_1,\dots,i_p}^{n=\dim\mathbb{X}^*}\frac{1}{p!}\mathbf{b}^{i_{p+1}}\wedge\dots\wedge\mathbf{b}^{i_n}\sqrt{g}\varepsilon_{i_1\dots i_p i_{p+1}\dots i_n}f^{i_1\dots i_p}\}$$

(iii) $$(*f)_{k_1\dots k_{n-p}}=\{\sum_{i_1,\dots,i_p}^{n=\dim\mathbb{X}^*}\sqrt{g}\varepsilon_{i_1\dots i_p k_1\dots k_{n-p}}f^{i_1\dots i_p}\}$$

as an element of the *skew algebra* $\mathbb{A}^{n-p}$ in the general associated cobasis $\{\mathbf{b}^{p+1}\wedge\dots\wedge\mathbf{b}^{i_n}\}$ with respect to the base space $\mathbf{x}\in\mathbb{X}\supset\mathbb{R}^n$ on dimension $n=\dim\mathbb{X}=\dim\mathbb{X}^*$ and $[g^{kl}]=\mathbf{G}^{-1}=\operatorname{adj}\mathbf{G}/\det\mathbf{G},\sqrt{g}=\sqrt{|g_{kl}|}$.

If we extend the algebra $\mathbb{A}^p$ of antisymmetric multilinear functions by $*1=\mathbf{e}^1\wedge\dots\wedge\mathbf{e}^n\in\mathbb{A}^n$ and $*\mathbf{e}^1\wedge\dots\wedge\mathbf{e}^n=1\in\mathbb{A}^0=\mathbb{R}$, respectively, let us collect some properties of *f, the dual of *f*.

Proposition 1-3 (**Hodge star operator,** the dual of an antisymmetric multilinear function):

Let the linearly ordered base $\{\mathbf{e}^1,\dots,\mathbf{e}^n\}$ *be orthonormal ("unimodular"). Then the Hodge star operator of an antisymmetric multilinear function f, the dual of f, with respect to* $\{\mathbf{e}^1,\dots,\mathbf{e}^n\}$ *satisfies the following:*

(i) *maps antisymmetric p-contravariant tensor-valued functions to anti-symmetric (n-p)-contravariant tensor-valued functions: $*:\mathbb{A}^p\to\mathbb{A}^{n-p}$

(ii) $$\begin{cases}*1=\mathbf{e}^1\wedge\dots\wedge\mathbf{e}^n=:\mathbf{E} & \textit{for every } 1\in\mathbb{A}^0,\mathbf{E}\in\mathbb{A}\\ *\mathbf{E}=1 & \textit{for every } 1\in\mathbb{A}^n,1\in\mathbb{A}^p\end{cases}$$

(iii) $**f=(-1)^{p(n-p)}f$ *for every* $f\in\mathbb{A}^p$

(iv $f\wedge *f=\|f\|^2\,\mathbf{e}^1\wedge\dots\wedge\mathbf{e}^n$ *with respect to the norm*

$$\|f\|^2=\frac{1}{p!}\sum_{i_1,\dots,i_p=1}^{n=\dim\mathbb{X}}f_{i_1\dots i_{p_1}}f^{i_1\dots i_p}$$

Example 1-4: Hodge star operator $n=\dim\mathbb{X}=\dim\mathbb{X}^*=3$, $\operatorname{span}\mathbb{X}^*=\{\mathbf{e}^1,\mathbf{e}^2,\mathbf{e}^3\},\mathbb{A}^p\to\mathbb{A}^{n-p}$

$n=3,p=0:\ *1=\mathbf{e}^1\wedge\mathbf{e}^2\wedge\mathbf{e}^3$

$$n=3,p=1:\quad *\mathbf{e}^{i_1}=\frac{1}{2}\varepsilon^{i_1}_{i_1 i_3}\mathbf{e}^{i_2}\wedge\mathbf{e}^{i_3}\quad\left[\begin{aligned}*\mathbf{e}^1&=\frac{1}{2}(\mathbf{e}^2\wedge\mathbf{e}^3-\mathbf{e}^3\wedge\mathbf{e}^2)=\mathbf{e}^2\wedge\mathbf{e}^3\\ *\mathbf{e}^2&=\frac{1}{2}(\mathbf{e}^3\wedge\mathbf{e}^1-\mathbf{e}^1\wedge\mathbf{e}^3)=\mathbf{e}^3\wedge\mathbf{e}^1\\ *\mathbf{e}^3&=\frac{1}{2}(\mathbf{e}^1\wedge\mathbf{e}^2-\mathbf{e}^2\wedge\mathbf{e}^1)=\mathbf{e}^1\wedge\mathbf{e}^2\end{aligned}\right.$$

$$n=3,p=2:\quad *\mathbf{e}^{i_1}\wedge\mathbf{e}^{i_2}=\mathbf{e}^{i_1 i_2}_{i_3}\quad\left[\begin{aligned}*\mathbf{e}^1\wedge\mathbf{e}^2&=\mathbf{e}^3\\ *\mathbf{e}^2\wedge\mathbf{e}^3&=\mathbf{e}^1\\ *\mathbf{e}^3\wedge\mathbf{e}^1&=\mathbf{e}^2\end{aligned}\right.$$

$n=3,p=3:\quad *\mathbf{e}^{i_1}\wedge\mathbf{e}^{i_2}\wedge\mathbf{e}^{i_3}=1$.

Example 1-5: Hodge star operator of an antisymmetric tensor-valued function, $n = \dim \mathbb{X} = \dim \mathbb{X}^* = 3$, $\mathbb{A}^p \to \mathbb{A}^{n-p}$

Throughout we apply the summation convention over repeated indices.

$$n=3,\ p=0: \begin{cases} f & \text{"}0-\textit{differential form}\text{"} \\ *f = f dx^1 \wedge dx^2 \wedge dx^3 & \text{"}3-\textit{differential form}\text{"} \end{cases}$$

$$n=3,\ p=1: \begin{cases} f = dx^{i_1} f_{i_1} & \text{"}1-\textit{differential form}\text{"} \\ *f = \frac{1}{2}\varepsilon^{i_1}_{i_2 i_3} dx^{i_2} \wedge dx^{i_3} f_{i_1} = & \\ = f_1 dx^2 \wedge dx^3 + f_2 dx^3 \wedge dx^1 + f_3 dx^1 \wedge dx^2 & \text{"}2-\textit{differential form}\text{"} \end{cases}$$

$$n=3,\ p=2: \begin{cases} f = \frac{1}{2} dx^{i_1} \wedge dx^{i_2} f_{i_1 i_2} & \text{"}2-\textit{differential form}\text{"} \\ *f = \frac{1}{2}\varepsilon^{i_1 i_2}_{i_3} \wedge dx^{i_3} f_{i_1 i_2} = & \\ = f_{23} dx^1 + f_{31} dx^2 + f_{12} dx^3 & \text{"}1-\textit{differential form}\text{"} \end{cases}$$

$$n=3,\ p=3: \begin{cases} f = \frac{1}{6} dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} f_{i_1 i_2 i_3} & \text{"}3-\textit{differential form}\text{"} \\ *f = \frac{1}{6}\varepsilon^{i_1 i_2 i_3} f_{i_1 i_2 i_3} = f_{123} & \text{"}0-\textit{differential form}\text{"} \end{cases}$$

♣

Example 1-6: Hodge star operator, $n = \dim \mathbb{X} = \dim \mathbb{X}^* = 3$, "$\times$"**product (cross product)**

By means of the *Hodge star operator* we are able to interpret the "$\times$" product ("cross product") in *three-dimensional* vector space. If the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{X}$, $\dim \mathbb{X} = 3$, presented in the orthonormal ("*unimodular*") base $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the following *equivalence* between $*\mathbf{x} \wedge \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$ holds:

$$\left.\begin{array}{l} \mathbf{x} = \mathbf{e}_i x^i, \mathbf{y} = \mathbf{e}_i y^i \,(\textit{summation convention}) \\ \mathbf{x} \in \mathbb{X}, \mathbf{y} \in \mathbb{Y}, \quad \mathrm{i}, \mathrm{j} \in \{1,2,3\} \end{array}\right] \Rightarrow$$

$$\mathbf{x} \wedge \mathbf{y} = \mathbf{e}_i \wedge \mathbf{e}_j x^i y^j = \mathbf{e}_1 \wedge \mathbf{e}_2 (x^1 y^2 - x^2 y^2) +$$
$$+ \mathbf{e}_2 \wedge \mathbf{e}_3 (x^2 y^3 - x^3 y^2) + \mathbf{e}_3 \wedge \mathbf{e}_1 (x^3 y^1 - x^1 y^3) \Rightarrow$$
$$\Rightarrow *(\mathbf{x} \wedge \mathbf{y}) = *(\mathbf{e}_i \wedge \mathbf{e}_i x^i y^j) = *(\mathbf{e}_1 \wedge \mathbf{e}_2)(x^1 y^2 - x^2 y^2) +$$
$$+ *(\mathbf{e}_2 \wedge \mathbf{e}_3)(x^2 y^3 - x^3 y^2) + *(\mathbf{e}_3 \wedge \mathbf{e}_1)(x^3 y^1 - x^1 y^3) =$$
$$= \varepsilon^k_{ij} \mathbf{e}_k x^i y^j$$

$$*(\mathbf{x}\wedge\mathbf{y}) = \mathbf{e}_3(x^1y^2 - x^2y^2) + \mathbf{e}_1(x^2y^3 - x^3y^2) + \mathbf{e}_2(x^3y^1 - x^1y^3) =$$
$$= \mathbf{e}_1(x^2y^3 - x^3y^2) + \mathbf{e}_2(x^3y^1 - x^1y^3) + \mathbf{e}_3(x^1y^2 - x^2y^2)$$
$$\left.\begin{array}{l}\mathbf{x}\times\mathrm{y} = \mathbf{e}_i\times\mathbf{e}_j x^i y^j \\ \mathbf{e}_i\times\mathbf{e}_j := \varepsilon_{ij}^k\mathbf{e}_k\end{array}\right] \Rightarrow \boxed{\mathbf{x}\times\mathbf{y} = *(\mathbf{x}\wedge\mathbf{y})}$$

By mean of the examples 1-5, 1-6 and 1-7 we like to make you familiar with (i) the *Hodge star operator* of an antisymmetric tensor-valued function over $\mathbb{R}^3$, (ii) its equivalence to the " $\times$ " product ("*cross product*") and (iii) *self-duality* in a *four-dimensional space*. Such a self-duality plays a key role in differential geometry and physics as being emphasized by M.F.Atiyah, N.J. Hitchin and J.M. Singer:

Example 1-7: Hodge star operator, $n = \dim \mathbb{X} = \dim \mathbb{X}^* = 4$,
$\mathbb{X} \in \mathbb{R}^4$ **, Minkowski space, self-duality**

(Atiyah, M.F.,Hitchin, N.J. and Singer, J. M.:
Self duality in four–dimensional Riemannian geometry.
Proc. Royal Soc. London *A362* (1978) 425-461)

Historical Aside

Thus we have constructed an *anticommutative algebra* by implementing the "*exterior product*" " $\wedge$ ", also called "*wedge product*", initiated by *H. Grassmann* in "*Ausdehnungslehre*" (second version published in 1882). See also his collected works, *H. Grassmann* (1911). In addition the work by *G. Peano* (Calcolo geometrico secondo, l`Ausdehnungslehre di Grassmann, *Fratelli Bocca Editori*, Torino 1888) should be mentioned here. This historical development may be documented by the work of *H. G. Forder* (1960). A modern version of the "*wedge product*" is given by *G. Berman* (1961). In particular we mention the contribution by *M. Barnabei et al* (1985) were by avoiding the notion of the dual $\mathbb{X}^*$ of a linear space $\mathbb{X}$ and based upon operations like *union, intersection, and complement*-i.e. known in *Boolean algebra*-have developed a double algebra with exterior products of type one ("*wedge product", "the join*") and of type two ("*the meet*"), namely "*to restore H. Grassmann's original ideas to full geometrical power*". The star operator "*" has been introduced by *W. V. D. Hodge*, being implemented into algebra in the work *W. V. D. Hodge* (1941) and *W. V. D. Hodge and D. Pedoe* (1978, p. 232-309). Here the start operation has been called "*dual Grassmann coordinates*"; in addition "*intersections and joins*" have been introduced.

Chapter 2
Linear Algebra

Multilinear algebra is built on linear algebra we are going into now. At first we give a careful definition of *linear algebra* which secondly we deepen by the diagrams "*Ass", "Uni*" and "*Comm*". The subalgebra "*ring with identity*" which is of central importance for solving polynomial equations by means of *Groebner bases,* the *Buchberger algorithm* and the *multipolynomial resultant* method is our third subject. Section four introduces the motion of division algebra and the non-associative algebra. Fifthly, we confront you with Lie algebra ("God is a lie Group"); in particular with *Witt algebra.* Section six compares *Lie algebra and Killing analysis.* Here we add some notes on the difficulties of a *composition algebra* in section seven. Finally in section eight *matrix algebra* is presented again, but this time as a *division algebra.* As examples of a division algebra as well as composition algebra we introduce *complex algebra* (*Clifford algebra* $C\ell(0,1)$) in section nine and *quaternion algebra* in section ten (*Clifford algebra* $C\ell(0,2)$) which is followed by an interesting letter of *W. R. Hamilton* (16 October 1943) to his son reproduced in section eleven. *Octonian algebra (Clifford algebra* with respect to $\mathbb{H}\times\mathbb{H}$) in section twelve is an example for a "non associative" algebra as well as a composition algebra. Of course, we have reserved "*section thirteen*" for the fundamental *Hurwitz theorem* of composition algebra and the fundamental *Frobenius theorem* of division algebra.

2-1 Definition of a Linear algebra

Up to now we have succeeded to introduce the *base space* $\mathbb{X}$ of vectors $\mathbf{x}\in\mathbb{X}=\mathbb{R}^3$ equipped with a metric and specialized to be *three dimensional Euclidean.* We have extended the base space to a tensor space, namely from vector-valued functions to tensor valued.

Definition 2-1 (linear algebra over the field of real numbers, linearity of vector space $\mathbb{X}$):

Let $\mathbb{R}$ be the field of real numbers. A <u>linear algebra</u> over $\mathbb{R}$ or $\mathbb{R}$-algebra consists of a set $\mathbb{X}$ of objects, two internal relations (either "<u>additive</u>" or "<u>multiplicative</u>") and one external relation

$$(\text{opera})_1 =: \alpha : \mathbb{X}\times\mathbb{X}\to\mathbb{X}$$

$$(\text{opera})_2 =: \beta : \mathbb{R}\times\mathbb{X}\to\mathbb{X} \text{ or } = \mathbb{X}\times\mathbb{R}\to\mathbb{X}$$

$$(\text{opera})_3 =: \gamma : \mathbb{X} \times \mathbb{X} \to \mathbb{X}\,.$$

[1] *With respect to the internal relation α ("join") $\mathbb{X}$ as a linear space is a vector space over $\mathbb{R}$, an Abelian group written "additively" or "multiplicatively":*

$$\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{X}$$

	additively written Abelian group		multiplicatively written Abelian group
	$a * (\mathbf{x}, \mathbf{y}) =: \mathbf{x} + \mathbf{y}$		$\alpha(\mathbf{x}, \mathbf{y}) =: \mathbf{x} \circ \mathbf{y}$
$(G1+)$	$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ (additive associativity)	$(G1\circ)$	$(\mathbf{x} \circ \mathbf{y}) \circ \mathbf{z} = \mathbf{x} \circ (\mathbf{y} \circ \mathbf{z})$ (multiplicative associativity)
$(G2+)$	$\mathbf{x} + 0 = \mathbf{x}$ (additive identity, neutral element)	$(G2\circ)$	$\mathbf{x} \circ \mathbf{1} = \mathbf{x}$ (multiplicative identity, neutral element)
$(G3+)$	$\mathbf{x} + (\text{-}\mathbf{x}) = 0$ (additive inverse)	$(G3\circ)$	$\mathbf{x} \circ \mathbf{x}^{-1} = \mathbf{1}$ (multiplicative inverse)
$(G4+)$	$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ (additive commutativity, Abelian axiom)	$(G4\circ)$	$\mathbf{x} \circ \mathbf{y} = \mathbf{y} \circ \mathbf{x}$ (multiplicative commutativity, Abelian axiom).

The triplet of axioms $\{(G1+), (G2+), (G3+)\}$ or $\{(G1\circ), (G2\circ), (G3\circ)\}$ constitutes the set of group axioms.

[2] *With respect to the external relation β the following compatibility conditions are satisfied*:

$$\mathbf{x}, \mathbf{y} \in \mathbb{X}\,,\ r, s \in \mathbb{R}$$

$$\beta(r, \mathbf{x}) =: r \times \mathbf{x}$$

$(D1+)$	$r \times (\mathbf{x} + \mathbf{y}) = (\mathbf{x} + \mathbf{y}) \times r$ $= r \times \mathbf{x} + r \times \mathbf{y} = \mathbf{x} \times r + \mathbf{y} \times r$ *(1st additive distributivity)*	$(D1\circ)$	$r \times (\mathbf{x} \circ \mathbf{y}) = (\mathbf{x} \circ \mathbf{y}) \times r$ $= (r \times \mathbf{x}) \circ \mathbf{y} = \mathbf{x} \circ (\mathbf{y} \circ r)$ *(1st multiplicative distributivity)*
$(D2+)$	$(r + s) \times \mathbf{x} = \mathbf{x} \times (r + s) =$ $= r \times \mathbf{x} + s \times \mathbf{x} = \mathbf{x} \times r + \mathbf{x} \times s$ *(2nd additive distributivity)*	$(D2\circ)$	$(r \circ s) \times \mathbf{x} = x \times r \circ s =$ $= r \times (s \times \mathbf{x}) = (\mathbf{x} \times r) \times s$ *(2nd multiplicative distributivity)*

$$(D3)\quad 1 \times \mathbf{x} = \mathbf{x} \times 1 = \mathbf{x}$$

(left and right identity)

[3] With respect to the internal relation γ ("meet") the following compatibility conditions are satisfied:

$$\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{X},\quad r \in \mathbb{R}$$

$$\gamma(\mathbf{x}, \mathbf{y}) =: \mathbf{x} * \mathbf{y}$$

$$(G1*)\quad (\mathbf{x} * \mathbf{y}) * \mathbf{z} = \mathbf{x} * (\mathbf{y} * \mathbf{z})$$

(*associativity w.r.t. internal multiplication*)

$$(D1*+)\quad \mathbf{x}*(\mathbf{y}+\mathbf{z})=\mathbf{x}*\mathbf{y}+\mathbf{x}*\mathbf{z}$$
$$(\mathbf{x}+\mathbf{y})*\mathbf{z}=\mathbf{x}*\mathbf{z}+\mathbf{y}*\mathbf{z}$$

(left and right additive distributivity w.r.t. internal multiplication)

$$(D1*\circ)\quad \mathbf{x}*(\mathbf{y}\circ\mathbf{z})=(\mathbf{x}*\mathbf{y})\circ\mathbf{z}$$
$$(\mathbf{x}\circ\mathbf{y})*\mathbf{z}=\mathbf{x}\circ(\mathbf{y}*\mathbf{z})$$

(left and right multiplicative distributivity w.r.t. internal multiplication)

$$(D2*\times)\quad r\times(\mathbf{x}*\mathbf{y})=(r\times\mathbf{x})*\mathbf{y}$$
$$(\mathbf{x}*\mathbf{y})\times r=\mathbf{x}*(\mathbf{y}r)$$

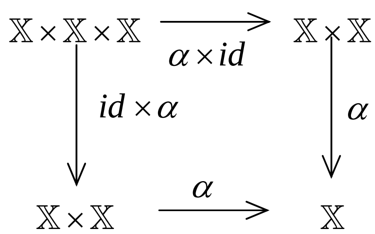
(left and right distributivity of internal and external multiplication)

2-2 The diagrams "*Ass*", "*Uni*" and "*Comm*"

Conventionally, a linear algebra is minimally constituted by the triplet $(\mathbb{X},\alpha,\beta)$ where $\mathbb{X}$ as a linear space is a vector space equipped with the linear maps $\alpha:\mathbb{X}\times\mathbb{X}\to\mathbb{X}$ and $\beta:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ satisfying the axioms (*Ass*) and (*Uni*) *according to the following diagrams:*

(Ass):

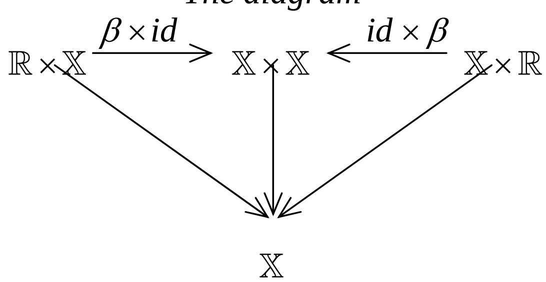
The square



commutes.

(Uni):

The diagram

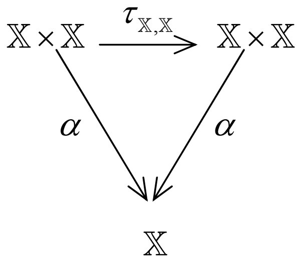


commutes.

Axiom (Ass) expresses the requirement that the multiplication α is *associative* whereas *Axiom* (*Uni*) means that the element $\beta(1)$ of $\mathbb{X}$ is a left as well as a right unit for α. The algebra $(\mathbb{X},\alpha,\beta)$ is commutative if in addition it satisfies the axiom

(Comm):

The triangle



commutes where $\tau_{\mathbb{X},\mathbb{X}}$ is the *flip* switching the factors: $\tau_{\mathbb{X},\mathbb{X}}(\mathbf{x} \circ \mathbf{y}) = \mathbf{y} \circ \mathbf{x}$. ♣

Indeed we have expressed a set of axioms both explicitly as well as in a diagrammatic approach which minimally constitute a linear algebra $(\mathbb{X}, \alpha, \beta)$. In addition, beside the first internal relation α called "*join*" we have experienced a second internal relation γ called "*meet*" which had to be made compatible with the other relations α and β, respectively. Actually, the diagram for the *axiom* (*Dis*) is left as an *exercise*.

Obviously we have experienced the words "*addition*" and "*multiplication*" for various binary operations. Note that in the linear algebra isomorphic to the vector space as its *geometric counterpart* we have *not* specified the inner multiplication $\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}) \in \mathbb{X}$. In a three-dimensional *vector space* of *Euclidean* type

$$\gamma(\mathbf{x}, \mathbf{y}) =: *(\mathbf{x} \wedge \mathbf{y}) = \mathbf{x} \times \mathbf{y},$$

namely the star * of the *exterior product* $\mathbf{x} \wedge \mathbf{y}$ or the "*cross product*" $\mathbf{x} \times \mathbf{y}$, for $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{y} \in \mathbb{R}^3$ is an *example*. Sometimes

$$\gamma(\mathbf{x}, \mathbf{y}) =: [\mathbf{x}, \mathbf{y}]$$

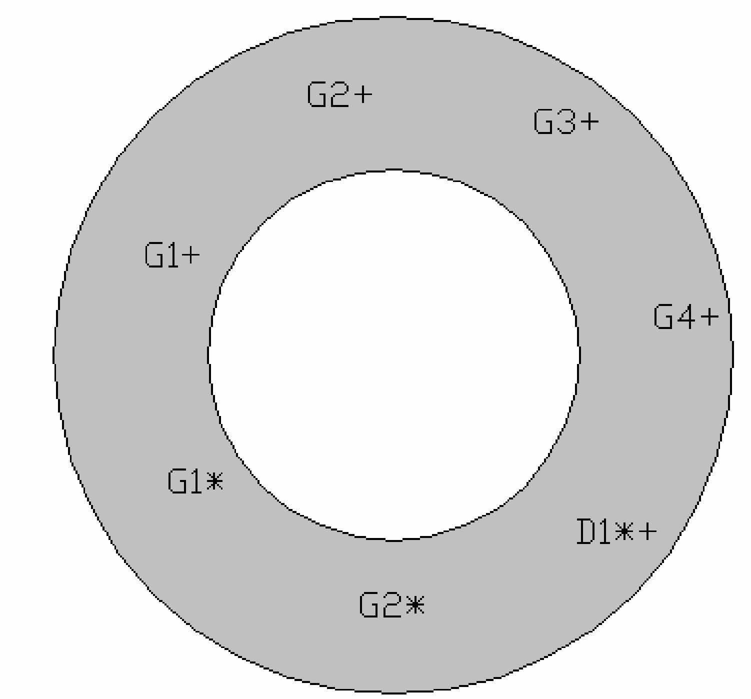
is written by rectangular brackets.

Historical Aside

Following a proposal of *L. Kronecker* ("Über die algebraisch auflösbaren Gleichungen (I. Abhandlung) *Monatsberichte der Akademie der Wissenschaften* 1853, Werke *4* (1929) 1-11) the axiom of *commutativity* $(G4+)$ *or* $(G4\circ)$ is called after *N. H. Abel* (Memoire sur un classe particuliere d'ecuations résolable algébraique, Crelle's J. reine angewandte Mathematik *4* (1828) 131-156 *Oeuvres* vol. 1, pages 478-514, vol. 2, pages 217-243, 329-331 edited by *S. Lie* and *L. Sylow*, Christiana 1881) who dealt with a particular class of equations of all degrees which are solvable by radicals, e.g. the cyclotomic equation $x^n - 1 = 0$. *N. H. Abel* has proven the following general theorem: If the roots of an equation are such that all roots can be expressed as rational functions of one of them, say *x*, and if any two of the roots, say $r_1 x$ and $r_2 x$ where r_1 and r_2 are *rational functions* are connected in such a way that $r_2\, r_1 x = r_1\, r_2 x$, then the equation can be solved by radicals. Refer $r_2\, r_1 x = r_1\, r_2 x$ to $(G1\cdot)$.

2-3 Ringed spaces: the subalgebra "ring with identity"

In $(G2\circ)$ the *neutral element* 1 as well as in $(G3\circ)$ the *inverse element* has been multiplied from the *right*. Similarly *left multiplication* $(G2\circ)$ by the neutral element 1 as well as $(G3\circ)$ by the inverse element are defined. Indeed it can be shown that there exist exactly one neutral element which is both *left-neutral* and right-neutral as well as exactly one inverse element which is both left-inverse and *right-inverse*. A subalgebra is called a "*ring with identity*" if the following *seven* conditions hold:



A ring with identity $(G3*)$ *is a division ring* if every nonzero element of the ring has a *multiplicative inverse*. A *commutative ring* is a ring with *commutative multiplication* $(G4*)$. *Modules* are generalizations of the vector spaces of linear algebra in which the "scalars" are allowed to be from an *arbitrary ring*, rather than a field of real numbers. They will be discussed as soon as we introduce *superalgebras*.

Now we take reference to

Lemma 2-2 (anticommutativity)

$\mathbf{x}\circ\mathbf{x}=0$ *for all* $\mathbf{x}\in\mathbb{X}\Leftrightarrow\mathbf{x}\circ\mathbf{y}=-\mathbf{y}\circ\mathbf{x}$ *for all* $\mathbf{x},\mathbf{y}\in\mathbb{X}$.

$"\circ"$ *is used in the notation* $"\wedge"$ *accordingly*.

:Proof:

$$"\Rightarrow"\qquad \mathbf{x}\circ\mathbf{y}+\mathbf{y}\circ\mathbf{x}=\mathbf{x}\circ\mathbf{x}+\mathbf{x}\circ\mathbf{y}+\mathbf{y}\circ\mathbf{x}+\mathbf{y}\circ\mathbf{y}=$$

$$=\mathbf{x}\circ(\mathbf{x}+\mathbf{y})+\mathbf{y}\circ(\mathbf{x}+\mathbf{y})=(\mathbf{x}+\mathbf{y})\circ(\mathbf{x}+\mathbf{y})=0$$

$$"\Leftarrow"\qquad \mathbf{x}=\mathbf{y}\Rightarrow\mathbf{x}\circ\mathbf{x}=-\mathbf{x}\circ\mathbf{x}\Rightarrow\mathbf{x}\circ\mathbf{x}=0$$

Lateron we refer to the following algebras.

2-4 Definition of a division algebra and non-associative algebra

Indeed we always have to invert a mapping to get something useful from identities. *Division algebra* is the proper algebraic tool to solve such problems.

Definition 2-3: (division algebra):

A $\mathbb{R}$-algebra is called *division algebra* over $\mathbb{R}$, if all non-null elements of $\mathbb{X}' := \mathbb{X}\backslash\{0\}$ form additionally a *group with respect to inner multiplication* $\mu =: \mathbf{x}\circ\mathbf{y}$ namely

$$\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{X}\backslash\{0\}$$

(*G*1∘) $(\mathbf{x}\circ\mathbf{y})\circ\mathbf{z} = \mathbf{x}\circ(\mathbf{y}\circ\mathbf{z})$ *(associativity of inner multiplication)*

(*G*2∘) $\mathbf{x}\circ 1 = \mathbf{x}$ *(identity of inner multiplication)*

(*G*3∘) $\mathbf{x}\circ\mathbf{x}^{-1} = 1$ *(inverse of inner multiplication)*

There are subalgebras which are classified as "non associative". Thus it may be better to have a precise definition of "*non-associative algebra*" at hand.

Definition 2-4: (non-associative algebra):

A weakening of a $\mathbb{R}$-algebra is the *non-associative algebra* over $\mathbb{R}$, if the axioms [1], [2] and [3] of a linear algebra *(Definition 2-1)* hold *with the exception* of (*G*1∘) that is the associativity of inner multiplication is cancelled.

2-5 Lie algebra, Witt algebra

"Perhaps God is a Lie group"

Many physicists believe that all modern physics is based on the operator algebra called "*Lie algebra*". Indeed more than 1000 textbooks are written on the subject. Indeed we can give it here a very short note, namely to be able lateron to compare *Lie algebra* and *Killing analysis*.

Definition 2-5 (*Lie algebra*):

A non associative algebra is called Lie algebra *over* $\mathbb{R}$*, if the following operations with respect to inner multiplication* $\mu =: \mathbf{x}\circ\mathbf{y}$ *hold*:

(L1) $\mathbf{x}\circ\mathbf{x} = 0$

(L2) $(\mathbf{x}\circ\mathbf{y})\circ\mathbf{z} + (\mathbf{y}\circ\mathbf{z})\circ\mathbf{x} + (\mathbf{z}\circ\mathbf{x})\circ\mathbf{y} = 0$ *(Jacobi identity)*

The examples of *Lie algebra* are numerous. As a special *Lie algebra* we present the *Witt algebra* which is applied to *Laurent polynomials*.

Example 2-1: Witt algebra on the ring of Laurent polynomials (Chen, Li, Math. Phys *167* (1995) 443 - 469):

The *Witt algebra W* is the complex Lie algebra of polynomials fields on the unit circle $\mathbb{S}^1$. An element of W is a linear combination of the elements of the form $\mathbf{e}^{in\Phi}(\partial/\partial\Phi)$, where Φ is a real parameter, and the *Lie bracket of W* is given by

$$\left[\mathbf{e}^{in\Phi}\frac{\partial}{\partial\Phi}, \mathbf{e}^{in\Phi}\frac{\partial}{\partial\Phi}\right] = i(n-m)\mathbf{e}^{i(m+n)}\frac{\partial}{\partial\Phi}$$

2-6 Lie algebra versus Killing analysis

Finally we switch to a comparison of *Lie algebra* and *Killing analysis.*

<u>2-6: Lie algebra versus Killing analysis</u>

Figure K2i:	Oblique parallel projection of the sphere S_r^2, coordinate lines $\mathbf{x}^1 = const$ ("meridians") versus $\mathbf{x}^2 = const$ ("parallel circles")
Figure K2ii:	Mercator projection of the sphere S_r^2, coordinate lines $\mathbf{x}^1 = const$ ("meridians") versus $\mathbf{x}^2 = const$ ("parallel circles")

Consider an *n*-dimensional *pseudo-Riemann* manifold $\{\mathbb{M}^n, g_{\mu\nu}(r,s)\}$, $n = r+s$, equipped with the *pseudo-Riemann metric* $g = g_{\mu\gamma}(\mathbf{x}^\lambda)d\mathbf{x}^\mu \vee d\mathbf{x}^\gamma$ represented by a local chart $x^\alpha \in \{\mathbb{R}^n, \delta_{\alpha\beta}(r,s)\}$ with *pseudo-Euclidean topology*. An *active transformation called* "act"

$$\mathbf{T}: x^\alpha \to x'^\alpha = f'^\alpha(x^\beta)$$

is a transformation of a point $p \in \mathbf{M}^n$ to another point $p' \in \mathbb{M}^n$ ("*point transformation"*) with respect to a *fixed chart.*

Example 2-2: $S_r^2(n=2,\ r=2,\ s=0)$, "*act*"

Move from one point $p \in S_r^2$ to another point $p' \in S_r^2$ along a great circle (*geodesic*) in the chart $\{x^1, x^2\}$ or {*spherical longitude, spherical latitude*}. Solution of an initial value problem of the differential equations of the *geodesic* ("*geodesic flow*") leads to *act* $\{x^1,x^2\}(p) -> \{x^1,x^2\}(p')$.

Alternative a *passive transformation* abbreviated by "*pass"*

$$\mathbf{T}: x^\alpha \to x^{\alpha'} = f^{\alpha'}(x^\beta)$$

also called "*cha-cha-cha"* or *change* from one *chart* to another *chart* is a transformation of *one fixed* point $p \in \mathbb{M}^n$ from the local chart $\{x^\alpha\}$ to another local chart $\{x\alpha'\}$. We refer to such a transformation $x^\alpha \to x^{\alpha'}$ as a "*Push forward*" and $x^{\alpha'} \to x^\alpha$ as a "*pull back"* operations.

Example 2-3: $S_r^2(n=2,\ r=2,\ s=0)$ *"pass"*

A transformation of the local chart $\{x^1, x^2\}$ of {*spherical longitude, spherical latitude*} into the alternative chart $\{x^{1'}, x^{2'}\}$ of *Mercator coordinates* (isometric coordinates, conformal coordinates) is given pointwise by

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \to \begin{bmatrix} x^{1'} \\ x^{2'} \end{bmatrix} = r \begin{bmatrix} x^1 \\ \ln\tan[\frac{\pi}{4} + \frac{1}{2}(x^2)] \end{bmatrix}.$$

This passive transformation is a *conformeomorphism* since it preserves the scalar product

$$g(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}) = g(\frac{\partial}{\partial x^{1'}}, \frac{\partial}{\partial x^{2'}})$$

For an illustration of such mappings let us refer you to Figure 2-1 and Figure 2-2. A more detailed example is

Example 2-4: (*pseudo-orthogonal group* **O**(1,1)):

Consider the *pseudo-orthogonal group* **O(**1,1**)** which is a *one-parameter group* known as the *Lorentz boost*

$$\mathbf{A}(\varepsilon) = \begin{bmatrix} \cosh\varepsilon & \sinh\varepsilon \\ -\sinh\varepsilon & \cosh\varepsilon \end{bmatrix}.$$

Correspondingly the *transformation group* is given by

$$\mathbf{x}' = \mathrm{f}(\mathbf{x}, \varepsilon) = \mathbf{A}(\varepsilon)\mathbf{x} \text{ subject to } \begin{bmatrix} \dim \mathbf{x}' = \dim \mathbf{x} = 2 \times 1 \\ \dim \mathbf{A} = 2 \times 2 \end{bmatrix}$$

or explicitly

$$\mathrm{f}(\mathbf{x}, \varepsilon) := \begin{bmatrix} x_{1'} \\ x_{2'} \end{bmatrix} = \begin{bmatrix} \mathrm{x}_1 \cosh\varepsilon - x_2 \sinh\varepsilon \\ \text{-}\mathrm{x}_1 \sinh\varepsilon + x_2 \cosh\varepsilon \end{bmatrix}$$

such that the "*Killing vector of symmetry*" is

$$\boldsymbol{\xi}(\mathbf{x}) = \frac{df}{d\varepsilon}(\varepsilon = 0) = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}$$

By the *exponential map* we may write the *Lie series*

$$x_{1'} = e^{\varepsilon Z} x_1 = x_1 + \varepsilon Z x_1 + \frac{\varepsilon^2}{2!} Z^2 x_1 + o(\varepsilon^3)$$

$$x_{2'} = e^{\varepsilon Z} x_2 = x_2 + \varepsilon Z x_2 + \frac{\varepsilon^2}{2!} Z^2 x_2 + o(\varepsilon^3)$$

introducing the *infinitesimal generator*

$$Z = -x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

which reproduces **O(1,1)** according to

$$Zx_1 = -x_2,\ \ Z^2 x_1 = Z(-x_2) = x_1,\ \ Z^3 x_1 = Zx_1 = -x_2 \ \text{ etc.}$$

$$Zx_2 = -x_1,\ \ Z^2 x_2 = Z(-x_1) = x_2,\ \ Z^3 x_2 = Zx_2 = -x_1 \ \text{ etc.}$$

$$x_{1'} = x_1 - \varepsilon x_2 + \frac{\varepsilon^2}{2!} x_1 - \frac{\varepsilon^3}{3!} x_2 + o(\varepsilon^4)$$

$$x_{2'} = x_2 - \varepsilon x_1 + \frac{\varepsilon^2}{2!} x_2 - \frac{\varepsilon^3}{3!} x_1 + o(\varepsilon^4)$$

which are *series expansions* of

$$\mathrm{x}_{1'} = \ \mathrm{x}_1 \cosh\varepsilon - x_2 \sinh\varepsilon,$$

$$\mathrm{x}_{2'} = -\mathrm{x}_1 \sinh\varepsilon + x_2 \cosh\varepsilon.$$

Notice the following definitions on symmetry transformation of type global *versus local*. In particular, we introduce the *Lie derivate* and *Lie differential* which leads us to the celebrated *Killing vector of symmetry*. *Killing analysis* is found on *Theorem K* which illustrated by *Example 2-6*: The symmetry transformation of the *Minkowski space* $\mathbb{E}^{3,1}$, namely the *inhomogeneous Lorentz group*, also called *Poincaré group* $P_{10}(\mathbb{R}^4)$ leads us to *Corollary K*.

Definition 2-6 (*global symmetry transformation*):

The transformation

$$\mathbf{T} : \mathbf{x} \to \mathbf{x}' = f(\mathbf{x})\,.$$

is called a *global symmetry transformation* (symmetry transformation in the large) of a *geometric object* **obj** if the geometric object does not *deform* under the transformation, in particular

$$\mathbf{obj}(p) = \mathbf{obj}(p') \ \text{ for all: } \ p' \in \mathbf{M}^n,\, p \in \mathbf{M}^n$$

Definition 2-7 (*local symmetry transformation*):

The infinitesimal transformation

$$\mathbf{T} : \mathbf{x} \to \mathbf{x}' = \lim_{\varepsilon \to} (\mathbf{x} + \varepsilon \boldsymbol{\xi}(\mathbf{x}))$$

is called a *local symmetry transformation* (symmetry transformation close to the identity) of a *geometric object* **obj** if the geometric object does not *deform* under the *infinitesimal transformation*, in particular

$$\partial_{\mathcal{L}}\mathbf{obj} := \lim_{\varepsilon\to 0}\frac{\mathbf{obj}(\mathbf{x}+e\boldsymbol{\xi}(\mathbf{x}))-\mathbf{obj}(\mathbf{x})}{\varepsilon}=0$$

versus

$$d_{\mathcal{L}}\mathbf{obj} := \lim_{\varepsilon\to 0}(\mathbf{obj}(\mathbf{x}+\varepsilon\boldsymbol{\xi}(\mathbf{x}))-\mathbf{obj}(\mathbf{x}))=0$$

$\partial_{\mathcal{L}}$ versus $d_{\mathcal{L}}$ are called the Lie derivate and the Lie differential, respectively. The vector

$$\boldsymbol{\xi}\in\{\mathbb{R}^n,\delta_{\alpha\beta}(r,s)\}$$

denotes the "*killing vector* $\boldsymbol{\xi}$ *of symmetry*".

Example 2-6: (*Killing analysis*, $\mathbf{E}^{3,1}$ ($n=4$, $r=3$, $s=1$) *Minkowski space, inhomogeneous Lorentz group, Poincaré group*):

$$g_{\mu\nu}\overset{*}{=}\delta^{-}_{\mu\nu}:=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix},\quad g_{\mu\nu,\lambda}\overset{*}{=}0$$

$$x^{\alpha}\in\{\mathbb{R}^4,\delta^{-}_{\alpha\beta}\}.$$

The *Killing equations* (K2i) are specialized into

$$\delta^{-}_{\rho\kappa}\delta_{\lambda}\xi^{\rho}+\delta^{-}_{\lambda\rho}\delta_{\kappa}\xi^{\rho}=0 \qquad \text{(Ex K3.1)}$$

or equivalently

$$\begin{bmatrix}\partial_{\mu}\xi^{\upsilon}+\partial_{\upsilon}\xi^{\mu}=0 & \text{for all } \mu\neq\upsilon\\ & \text{(no summation over } \mu,\upsilon)\\ \partial_{\mu}\xi^{\mu}=0 & \text{for all } \mu\\ & \text{(no summation over } \mu)\end{bmatrix} \qquad \text{(Ex K3.2)}$$

1st step : differentiate the first equation

$$\partial_{\lambda}\partial_{\mu}\xi^{\upsilon}+\partial_{\lambda}\partial_{\mu}\xi^{\mu}=0 \text{ for all } \lambda,\mu,\gamma \text{ (\textit{no summation})} \qquad \text{(Ex.K3.3)}$$

2nd step : integrability condition

$$\boldsymbol{\xi}^{\mu}_{\lambda\upsilon}=\boldsymbol{\xi}^{\mu}_{\upsilon\lambda}. \qquad \text{(Ex K3.4)}$$

Once we combine both the steps we are led to

$$\frac{\partial^2 \xi^\lambda}{\partial x^\mu \partial x^\upsilon} = 0 \qquad \text{(Ex K3.5)}$$

with the *general solution*

$$\xi^\lambda = \delta\omega_\mu^\lambda x^\mu + a^\lambda \ \ versus \ \ \xi = \delta\mathbf{\Omega}\mathbf{x} + \mathbf{a} \qquad \text{(Ex K3.6)}$$

$$\frac{\partial \xi^\lambda}{\partial x^\mu} = \delta\omega_\mu^\lambda \qquad \text{versus} \qquad \text{grad}\boldsymbol{\xi} = \delta\mathbf{\Omega}$$

$$\frac{\partial^2 \xi^\lambda}{\partial x^\mu \partial x^\upsilon} = 0 \qquad \text{versus} \qquad \text{grad} \otimes \text{grad}\boldsymbol{\xi} = 0.$$

(Ex K3.1) and (Ex K3.2) imply

$$\left.\begin{array}{l} \delta\omega_\mu^\upsilon + \delta\omega_\upsilon^\mu = 0 \text{ for all } \mu \neq \upsilon \\ \delta\omega_\mu^\mu = 0 \text{ for all } \mu\, (no\ summation\ over\ repeated\ indices) \end{array}\right] \Leftrightarrow$$

$$\delta\mathbf{\Omega} = -\delta\mathbf{\Omega}^T$$

Obviously the matrix $\delta\mathbf{\Omega}$ is *antisymmetric*. Finally we collect the results in

Corollary K: $\mathbf{E}^{3,1}$, *inhomogeneous Lorentz group, Poincaré group* $P_{10}(\mathbb{R}^4)$

The infinitesimal transformation which leaves the metric g of a Minkowski space undeformed is the infinitesimal inhomogeneous Lorentz group of transformations (Poincaré group) with six (pseudo-)rotational parameters and four translational parameters (in toto $r = 10$ *parameters)*

$$\delta\mathbf{x} := \mathbf{x}' - \mathbf{x} = \delta\mathbf{\Omega}\mathbf{x} + \mathbf{a}$$

which can be transformed globally into

$$\mathbf{x}' = \mathbf{\Lambda}\mathbf{x} + \mathbf{a} \ \ subject\ to\ \ \mathbf{\Lambda}^T\mathbf{I}^-\mathbf{\Lambda} = \mathbf{I}^- or\ \ \delta\mathbf{\Lambda}\mathbf{\Lambda}^{-1} = \delta\mathbf{\Omega}\,.$$

♣

Theorem K: (local symmetry transformation of the metric: *isometry*, **Killing equations**)

In order that a local symmetry transformation of the metric g (local isometry) exists, it is necessary and sufficient that the Lie derivative of g vanishes, in particular

$$\partial_{\mathcal{L}} g = 0\ . \qquad \text{(K2)}$$

The zero identity of the Lie derivative of the metric is represented alternatively by the Killing equations

$$K\boldsymbol{\xi} = 0 \Leftrightarrow \qquad \text{(K2)}$$

$$\Leftrightarrow \quad (i) \quad \xi^\mu \partial_\mu g_{\lambda\kappa} + g_{\rho\kappa}\partial_\lambda \xi^\rho + g_{\lambda\rho}\partial_\kappa \xi^\rho = 0 \ \ or \qquad \text{(K2i)}$$

$\Leftrightarrow$	*(ii)*	$\xi^\mu \partial_\mu g_{\lambda\kappa} + 2g_{(\rho\|\kappa)} \partial_\lambda \xi^\rho = 0$ *or*	(K2ii)
$\Leftrightarrow$	*(iii)*	$2\xi_{(\mu;\gamma)} = 0$	(K2iii)

As differential equations for the Killing vector $\boldsymbol{\xi}$ *they are subject to the integrability conditions*

(iv)	$\dfrac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\beta} = \dfrac{\partial^2 \xi^\mu}{\partial x^\beta \partial x^\alpha}$	*or*	(K3i)
(v)	$\xi_{\alpha;\beta;\gamma} = -\xi_\sigma R^\sigma_{\gamma\alpha\beta}$	*or*	(K3ii)
(vi)	$\xi^\rho R_{\nu\mu\lambda\kappa;\rho} - \xi_{\rho;\sigma}(R^{\cdot\cdot\cdot\sigma}_{\nu\mu\lambda}\delta^\rho_\kappa - R^{\cdot\cdot\cdot\rho}_{\kappa\lambda\mu}\delta^\rho_\nu - R^{\cdot\cdot\cdot\rho}_{\lambda\kappa\nu}\delta^\sigma_\mu + R^{\cdot\cdot\cdot\rho}_{\nu\mu\kappa}\delta^\sigma_\lambda) = 0$		(K3iii)

where $R^{\cdot\cdot\cdot\rho}_{\lambda k\nu}$ are the covariant and contravariant coordinates of the *Riemann curvature tensor* "Riemann (3,1)"

Proof:

1st step $g \to g'$ by "pass" $x^{\mu'} = \delta^{\mu'}_\alpha [x^\alpha + \varepsilon \xi^\alpha (x^\beta)]$

(*transformation close the identity*) $\Rightarrow dx^{\mu'} = \delta^{\mu'}_\alpha \left[\delta^\alpha_\mu + \varepsilon \partial_\mu \xi^\alpha (x^\beta) \right] dx^\mu$

(infinitesimal transformation close to the identity *or* "*cha-cha-cha*" or "*pullback*"

2nd step Choose the *metric* as the geometric object **obj** under consideration:

$$\left.\begin{array}{c} ds'^2 = g_{\mu'\nu'}(x^{\lambda'}) dx^{\mu'} dx^{\nu'} \\ g_{\mu'\nu'}(x^{\lambda'}) = g_{\mu'\nu'}(x^\alpha + \varepsilon \xi^\alpha (x^\beta)) = \delta^\beta_{\mu'} [g_{\alpha\beta}(x^\kappa) + \varepsilon \xi^\mu \partial_\mu g_{\alpha\beta}(x^\kappa) + o(\varepsilon^2)] \end{array}\right] \Rightarrow$$

$$ds'^2 = [g_{\mu\nu} + \varepsilon \{\xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\alpha\nu} \partial_\mu \xi^\alpha + g_{\mu\beta} \partial_\nu \xi^\beta + o(\varepsilon^2)\}] dx^\mu dx^\nu$$

3rd step

$$\left.\begin{array}{c} g = g' \\ g \to g' \end{array}\right] \Rightarrow \left.\begin{array}{c} ds^2 = ds'^2 \\ ds'^2 \ \ of \ \ above \end{array}\right] \Rightarrow \xi^\lambda d_\lambda g_{\mu\nu} + g_{\alpha\nu} \partial_\mu \xi^\alpha + g_{\mu\beta} \partial_\nu \xi^\beta = 0$$

end

The statements (ii) and (iii) follow directly from the symmetric permutation $(ab) = (ab + ba)/2$ as well as from the definition of the *covariant derivative* written by a semicolon and *the Ricci Lemma* $g_{\mu\nu;\lambda} = 0$. For geodetic use we present to you the *Killing analysis* of the sphere S^2_r in *Example* 2-7 and of the ellipsoid of revolution $\mathbb{E}^2_{a,b}$ in *Example* 2-8. Finally we refer to related papers on the *conformal group* $\mathbb{C}$ and its related Killing analysis which is basic for conformal mapping (conformal = morphism) and *conformal field theory.*

Four remarks have to be made with respect to the definition of a symmetry transformation which is based upon zero deformation of a geometric object under the action or the passivity of a symmetry transformation. *Firstly,* we have to reflect the notion of "symmetry". Here the intuition is taking reference to the idea that a geometric object does not change under a transformation if there is some symmetry. *Secondly,* the notion of *deformation* is known in topology. Alternative notions for the statement that a geometric object dos not deform under the action or the passivity of a symmetry transformation are the following: A geometric object is *equivariant* or *covariant with respect to a symmetry transformation* or *form invariant. Thirdly,* the symmetry transformations are more precisely called

transformation groups

Following the axioms ($G1$), ($G2$), ($G3$) of a *non-Abelian group* of Appendix A. That is the group axioms apply *without the axiom of commutativity* ($G4$), in general. Let us denote the algebraic binary operation "$\circ$" applied to the transformations f_{I} and f_{II} respectively. If f_{I} as well as f_{II} are elements of a group, then by composition of functions $f_{\mathrm{I}} \circ f_{\mathrm{II}}$ is an element of the group, a relation we identify as the *axiom of closure*. In addition, if f_{I}, f_{II}, f_{III} are *elements of the group,* than the group axioms hold.

$$(G1\circ)\quad f_I \circ (f_{II} \circ f_{III}) = (f_I \circ f_{II}) \circ f_{III} \qquad (\textit{associativity})$$
$$(G2\circ)\quad id \circ f = f \circ id = f \qquad (\textit{identity})$$
$$(G3\circ)\quad f_I \circ f_{II} = f_{II} \circ f_I = id \Rightarrow f_{II} = f_I^{-1} \qquad (\textit{inverse})$$

Fourthly the transformations groups are considered as *differential manifolds* $\mathbf{M}^r$ of dimension *r*. The charts which cover the differential manifolds $\mathbf{M}^r$ are based on *r* coordinates which are called the *parameters* of the transformation group. *For instance,* if the transformation group is a proper rotation in a two-dimensional Euclidean space, namely an element of ***SO2***, then the proper rotation matrix generates the *one-dimensional manifold* of type $\boldsymbol{S}^1$, a circle. The one parameter is the rotation angle. In this sense, a local symmetry transformation is called

r-parametric, local Lie transformation group.

2-7 Definition of a composition algebra

Various algebras are generated by adding an additional structure to the minimal set of axioms of a linear algebra blocked by 1, 2 and 3. A special example is given by

Definition 2-8 (composition algebra):

A *non-associative algebra* with 1 as identity of inner multiplication is called *composition algebra* over $\mathbb{R}$, if there exists a regular quadratic form $Q : X \to \mathbb{R}$ which is compatible with the corresponding operations that is the following operation hold:

(K1) $Q : X \to \mathbb{R}$ *is a regular quadratic form,*

$\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}, \quad r \in \mathbb{R}$

$Q(r \times \mathbf{x}) = r^2 \times Q(\mathbf{x}) \quad (quadratic form)$

$Q(\mathbf{x}+\mathbf{y}+\mathbf{z}) = Q(\mathbf{x}+\mathbf{y}) + Q(\mathbf{x}+\mathbf{z}) + Q(\mathbf{y}+\mathbf{z}) - Q(\mathbf{x}) - Q(\mathbf{y}) - Q(z)$

$Q(r \times \mathbf{x}+\mathbf{y}) - r \times Q(\mathbf{x}+\mathbf{y}) = (r-1) \times [r \times Q(\mathbf{x}) - Q(\mathbf{y})]$

$Q(\mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0 \quad (regularity)$

(K2) $Q(\mathbf{x} \Lambda \mathbf{y}) = Q(\mathbf{x} \times Q(\mathbf{y}) \quad (multiplicativity)$

$Q(\mathbf{1}) = 1$

The quadratic form introduced by Definition 2-8 leads to the topological notion of scalar products, norm and metric we already used:

Lemma 2-9 (scalar product, norm, metric):

In a composition algebra with a positive-definite quadratic form a *scalar product* ("inner product") is defined by the bilinear form $< \cdot | \cdot >: \mathbf{X} \times \mathbf{X}^*$ $\to \mathbb{R}$ with

$$< \mathbf{x} | \mathbf{y} > := \frac{1}{2}[Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})];$$

a *norm* is defined by $\| \cdot \|: \mathbf{X} \to \mathbb{R}$ with

$$\| \mathbf{x} \| := +[Q(\mathbf{x})]^{1/2}$$

and *metric* is defined by the bilinear form

$$\rho(\mathbf{x}, \mathbf{y}) := +[Q(\mathbf{x}-\mathbf{y})]^{1/2} .$$

Thus to the *algebraic structure a topological* structure is added, if in addition

(K3) $Q(\mathbf{x}) \geq 0$

for all $\mathbf{x} \in X$ holds.

Proof:

(i) scalar product

$< \cdot | \cdot >: X \times X \to \mathbb{R}$ *is a scalar product since*

(1) $$< \mathbf{x} | \mathbf{y} > = \frac{1}{2}[Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})] =$$

$$= \frac{1}{2}[Q(\mathbf{y}+\mathbf{x}) - Q(\mathbf{y}) - Q(\mathbf{x})] = < \mathbf{y} | \mathbf{x} > \quad (symmetry)$$

(2) $$< \mathbf{x}+\mathbf{y} | \mathbf{z} > = \frac{1}{2}[Q(\mathbf{x}+\mathbf{y}+\mathbf{z}) - Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{z})] =$$

$$= \frac{1}{2}[Q(\mathbf{x}+\mathbf{z}) - Q(\mathbf{x}) - Q(\mathbf{z})] +$$

$$+ \frac{1}{2}[Q(\mathbf{y}+\mathbf{z}) - Q(\mathbf{y}) - Q(\mathbf{z})] =$$

$$= <\mathbf{x} \mid \mathbf{z}> + <\mathbf{y} \mid \mathbf{z}> \quad (\textit{additivity})$$

(3) $$<r\mathbf{x} \mid \mathbf{y} = \frac{1}{2}[Q(r\mathbf{x}+\mathbf{y}) - Q(r\mathbf{x}) - Q(\mathbf{y})] =$$

$$= \frac{1}{2} r[Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})] = r <\mathbf{x} \mid \mathbf{y}> \quad (\textit{homogeneity})$$

(4) $$<\mathbf{x} \mid \mathbf{x}> = \frac{1}{2}[Q(\mathbf{x}+\mathbf{x}) - Q(\mathbf{x}) - Q(\mathbf{x})] =$$

$$= \frac{1}{2}[Q(2\mathbf{x}) - 2Q(\mathbf{x})] = Q(\mathbf{x}) \geq 0 \quad (\textit{positivity})$$

(ii) norm

$\|\cdot\| : \mathbf{X} \to \mathbb{R}$ *is a norm since*

(N1) $\|\mathbf{x}\| := +\left[Q(\mathbf{x})\right]^{1/2} \geq 0$ (*positivity*)

and

$\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$

(N2) $\|r\mathbf{x}\| = +\left[Q(r\mathbf{x})\right]^{1/2} = +\left[r^2 Q(\mathbf{x})\right]^{1/2} = |r| \times \|\mathbf{x}\|$ (*homogeneity*)

(N3) $\|\mathbf{x}+\mathbf{y}\| = +\left[Q(\mathbf{x}+\mathbf{y})\right]^{1/2} = +\left[Q(\mathbf{x}) + Q(\mathbf{y}) + 2 <\mathbf{x} \mid \mathbf{y}>\right]^{1/2} =$

$$= +[\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\| \cdot \|\mathbf{y}\|]^{1/2} \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

(Cauchy-Schwarz' inequality) "triangle inequality"

(iii) metric

$\rho : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ *is a metric since*

(M1) $\rho(\mathbf{x},\mathbf{y}) = +[Q(\mathbf{x}-\mathbf{y})]^{1/2} = \|\mathbf{x}-\mathbf{y}\| \geq 0$ (*positivity*)

and

$\rho(\mathbf{x},\mathbf{y}) = \mathbf{0} \Leftrightarrow \mathbf{x}-\mathbf{y} = \mathbf{0} \Leftrightarrow \mathbf{x} = \mathbf{y}$

(M2) $\rho(\mathbf{x}, y) = \|\mathbf{x}-\mathbf{y}\| = |-1| \cdot \|\mathbf{y}-\mathbf{x}\| = \rho(\mathbf{y},\mathbf{x})$ (*symmetry*)

(M3) $\rho(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \|(\mathbf{x}-\mathbf{z}) + (\mathbf{z}-\mathbf{y})\| \leq$

$$\leq \|\mathbf{x}-\mathbf{z}\| + \|\mathbf{z}-\mathbf{x}\| = \rho(\mathbf{x},\mathbf{z}) + \rho(\mathbf{z},\mathbf{y}) \quad (\textit{triangle inequality})$$ ♣

2-8 Matrix algebra as a division algebra

While matrix algebra has been presented so far more intuitively with respect to linear algebra we shall deviate this section exclusively to matrix algebra as a division algebra over the field of real numbers.

Example 2-8: Matrix algebra as a division algebra

$$\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n\times m}$$

1 $\mathbf{A},\mathbf{B},\mathbf{C} \in \mathbb{R}^{nm}, \quad \alpha(\mathbf{A},\mathbf{B}) =: \mathbf{A}+\mathbf{B}$

$$(G1+) \quad (\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$$
$$(G2+) \quad \mathbf{A}+0 = \mathbf{A}$$
$$(G3+) \quad \mathbf{A}-\mathbf{A} = 0$$
$$(G4+) \quad \mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$$

2 $\mathbf{A},\mathbf{B} \in \mathbb{R}^{nm}, \; r,s \in \mathbb{R}, \; \beta(r,\mathbf{A}) =: r\times\mathbf{A}$

$$(D1+) \quad r\times(\mathbf{A}+\mathbf{B}) = r\times\mathbf{A}+r\times\mathbf{B}$$
$$(D2+) \quad (r+s)\times\mathbf{A} = r\times\mathbf{A}+s\times\mathbf{A}$$
$$(D3+) \quad 1\times\mathbf{A} = \mathbf{A}$$

3 "multiplication of matrices"

(i) "Cayley-product" (just "the matrix product")

$$\left.\begin{matrix}\mathbf{A}=[a_{ij}]\in\mathbb{R}^{n\times\ell}, & \dim\mathbf{A}=n\times\ell \\ \mathbf{B}=[b_{ij}]\in\mathbb{R}^{\ell\times m}, & \dim\mathbf{B}=\ell\times m\end{matrix}\right] \Rightarrow \mathbf{C}:=\mathbf{A}\cdot\mathbf{B}=[c_{ij}]\in\mathbb{R}^{nm},$$

$$\dim\mathbf{C}=n\times m \qquad c_{ij}:=\sum_{k=1}^{\ell} a_{ik}b_{k\ell}.$$

The product was introduced by *A. Cayley*: A memoir on the theory of matrices, Phil. Trans. Roy. Soc. London *148* (1857) 17-37; see also his Collected Works, Vol. 2, 475-496. A historical perspective is given in *R. W. Feldmann:* Matrix theory I: Arthur Cayley-founder of matrix theory, Mathematics Teacher *57* (1962) 482-484.

(ii) "Kronecker-Zehfuß-product"

$$\left.\begin{matrix}\mathbf{A}=[a_{ij}]\in\mathbb{R}^{n\times m}, & \dim\mathbf{A}=n\times m \\ \mathbf{B}=[b_{ij}]\in\mathbb{R}^{k\times\ell}, & \dim\mathbf{B}=k\times\ell\end{matrix}\right] \Rightarrow \mathbf{C}:=\mathbf{B}\otimes\mathbf{A}=[c_{ij}]\in\mathbb{R}^{kn\times\ell m},$$

$$\dim\mathbf{C}=kn\times\ell m, \qquad \mathbf{B}\otimes\mathbf{A}:=[b_{ij}\mathbf{A}].$$

The product was early referenced to *L. Kronecker* by *C. C. MacDuffee:* The theory of matrices (1933), reprint Chelsea Publ., New York 1946.

The other reference is *J. G. Zehfuss:* Über eine gewisse Determinante, Z. Mat. Phys. *3* (1858) 296-301. See also *H. V. Hendersson, F. Pukelsheim and S. R. Searle:* On the history of the Kronecker product, Linear and Multilinear Algebra *14* (1983) 133-120 and *R. A. Horn and C. R. Johnson:* Topics in matrix analysis, chapter four, Cambridge University Press, Cambridge 1991. In order to discriminate the *tensor product* $\mathbf{A}\otimes\mathbf{B}$ of two tensors ***A*** and ***B*** from its *matrix* representation, J. Dauxois et al. (1994) propose the notation $M(\mathbf{A}\otimes\mathbf{B})=\mathbf{A}\overset{K}{\otimes}\mathbf{B}$ for the matrix representation of the tensor product where "$\overset{K}{\otimes}$"emphasizes the *Kronecker-Zehfuss product.*

(iii) "Khatri-Rao-product" (two rectangular matrices of identical column number)

$$\left.\begin{array}{ll}\mathbf{A}=[\mathbf{a}_i,...,\mathbf{a}_m]\in\mathbb{R}^{n\times m}, & \dim\mathbf{A}=n\times m\\ \mathbf{B}=[\mathbf{b}_i,...,\mathbf{b}_m]\in\mathbb{R}^{k\times m}, & \dim\mathbf{B}=k\times m\end{array}\right]\Rightarrow$$

$$\Rightarrow\mathbf{C}:=\mathbf{B}\odot\mathbf{A}:=[\mathbf{b}_1\otimes\mathbf{a}_i,...,\mathbf{b}_m\otimes\mathbf{a}_m]\in\mathbb{R}^{kn\times m}$$

$$\dim\mathbf{C}=kn\times m\,.$$

Their product was introduced by *C. G. Khatri and C. R. Rao:* Solutions to some fundamental equations and their applications to characterization of probability distributions, Sankya *A30* (1968) 167-180.

(iv) "Hadamard product" (two rectangular matrices of the same dimension, element-wise product)

$$\left.\begin{array}{ll}\mathbf{G}=\left[g_{ij}\right]\in\mathbb{R}^{n\times m}, & \dim\mathbf{G}=n\times m\\ \mathbf{H}=\left[h_{ij}\right]\in\mathbb{R}^{m\times m}, & \dim\mathbf{H}=n\times m\end{array}\right]\Rightarrow\mathbf{K}:=\mathbf{G}*\mathbf{H}=\left[k_{ij}\right]\in\mathbb{R}^{n\times m}$$

$$\dim\mathbf{K}=n\times m,\ \ k_{ij}:=g_{ij}h_{ij}\ \ (\textit{no summation}).$$

The product was introduced by *J. Hadamard:* Theoreme sur les series entieres, Acta Math. *22* (1899) 1-28. See also *Th. Moutard:* Notes sur les equations derivees partielles, J. de l'Ecole Polytechnique *64* (1894) 55-69, as well as *J. Hadamard:* Lecons sur la propagardion des ondes et les equations de l'hydrodynamique, Paris 1893, reprint Chelsea Publ., New York 1949 and *I. Schur:* Bemerkungen zur Theorie der verschrankten Bilinearformen mit unendlich vielen Veraenderlichen, J. Reine und Angew. Math. *140* (1911) 1-28 and *R. A. Horn and C. R. Johnson:* Topics in matrix analysis, chapter five, Cambridge University Press, Cambridge 1991.

In general the existence of the *Cayley product* $\mathbf{A}\cdot\mathbf{B}$ does not imply the existence of the *Cayley product* $\mathbf{B}\cdot\mathbf{A}$. If both products exist, they are not equal in general. Two quadratic matrices **A** and **B** satisfying

$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ are called commutative. A numerical example of the various products is the following:

(i)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \boldsymbol{Z}^{2\times3}, \mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} \in \boldsymbol{Z}^{3\times2} \Rightarrow$$

$$\Rightarrow \mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 28 & 34 \\ 64 & 79 \end{bmatrix} \in \boldsymbol{Z}^{2\times2} \text{ ("integer numbers").}$$

(ii)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \boldsymbol{Z}^{2\times3}, \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \in \boldsymbol{Z}^{3\times2} \Rightarrow \mathbf{B} \otimes \mathbf{A} = \left[b_{ij} \cdot \mathbf{A} \right] =$$

$$= [\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \otimes \mathbf{A}] = \begin{bmatrix} 1 & 2 & 3 & 2 & 4 & 6 \\ 4 & 5 & 6 & 8 & 10 & 12 \\ 3 & 6 & 8 & 4 & 8 & 12 \\ 12 & 15 & 18 & 16 & 20 & 24 \\ 5 & 10 & 15 & 6 & 12 & 18 \\ 20 & 25 & 30 & 24 & 30 & 36 \end{bmatrix} \in \boldsymbol{Z}^{6\times6}$$

(iii)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \boldsymbol{Z}^{2\times3}, \ \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \in \boldsymbol{Z}^{2\times3} \Rightarrow$$

$$\Rightarrow \mathbf{B} \odot \mathbf{A} = [\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 6 \end{bmatrix}] =$$

$$= \begin{bmatrix} 1 & 4 & 9 \\ 4 & 10 & 18 \\ 4 & 10 & 18 \\ 16 & 25 & 36 \\ 7 & 16 & 27 \\ 23 & 40 & 54 \end{bmatrix} \in \boldsymbol{Z}^{6\times3} \text{ ("integer numbers")}$$

(iv)

$$\mathbf{G} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \boldsymbol{Z}^{2\times3}, \quad \mathbf{H} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \in \boldsymbol{Z}^{2\times3} \Rightarrow$$

$$\Rightarrow \mathbf{G} * \mathbf{H} = \left[g_{ij} h_{ij} \right] = \begin{bmatrix} 2 & 6 & 12 \\ 20 & 30 & 42 \end{bmatrix} \in \boldsymbol{Z}^{2\times3}$$

(i1)

$(G1\cdot)\quad (\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}=\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$

$(D1\cdot+)\quad \mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C},\ (\mathbf{A}+\mathbf{B})\cdot\mathbf{C}=\mathbf{AC}+\mathbf{BC}$

$(D2\cdot\times)\quad r\times(\mathbf{A}\cdot\mathbf{B})=(r\times\mathbf{A})\cdot\mathbf{B},\quad (\mathbf{A}\cdot\mathbf{B})^T=\mathbf{B}^T\mathbf{A}^T$

(*ii*1)

$(G1\otimes)\quad (\mathbf{A}\otimes\mathbf{B})\otimes\mathbf{C}=\mathbf{A}\otimes(\mathbf{B}\otimes\mathbf{C})=\mathbf{A}\otimes\mathbf{B}\otimes\mathbf{C}$

$(D1\otimes+\ell)\quad (\mathbf{A}+\mathbf{B})\otimes\mathbf{C}=\mathbf{A}\otimes\mathbf{C}+\mathbf{B}\otimes\mathbf{C})$

$(D1\otimes+r)\quad \mathbf{A}\otimes(\mathbf{B}+\mathbf{C})=(\mathbf{A}\otimes\mathbf{B})+(\mathbf{A}\otimes\mathbf{C})$

$(D2\otimes\times)\quad r\times(\mathbf{A}\otimes\mathbf{B})=(r\times\mathbf{A})\otimes\mathbf{B}$

$$(\mathbf{A}\otimes\mathbf{B})\cdot(\mathbf{C}\otimes\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})\otimes(\mathbf{B}\cdot\mathbf{D})$$

$$(\mathbf{A}\otimes\mathbf{B})^T=\mathbf{B}^T\otimes\mathbf{A}^T$$

(iii1)

$(G1\odot)\quad (\mathbf{A}\odot\mathbf{B})\odot\mathbf{C}=\mathbf{A}\odot(\mathbf{B}\odot\mathbf{C})=\mathbf{A}\odot\mathbf{B}\odot\mathbf{C}$

$(D1\odot+l)\quad (\mathbf{A}+\mathbf{B})\odot\mathbf{C}=\mathbf{A}\odot\mathbf{C}+\mathbf{B}\odot\mathbf{C}$

$(D1\odot+r)\quad \mathbf{A}\odot(\mathbf{B}+\mathbf{C})=\mathbf{A}\odot\mathbf{B}+\mathbf{A}\odot\mathbf{C}$

$(D2\odot\times)\quad r\times(\mathbf{A}\odot\mathbf{B})=(r\times\mathbf{A})\otimes\mathbf{B}$

$$(\mathbf{A}\cdot\mathbf{C})\odot(\mathbf{B}\cdot\mathbf{D})=(\mathbf{A}\otimes\mathbf{B})\cdot(\mathbf{C}\odot\mathbf{D})$$

(iv1)

$(G4*)\quad \mathbf{A}*\mathbf{B}=\mathbf{B}*\mathbf{A}$

$(G1*)\quad (\mathbf{A}*\mathbf{B})*\mathbf{C}=\mathbf{A}*(\mathbf{B}*\mathbf{C})=\mathbf{A}*\mathbf{B}*\mathbf{C}$

$(D1*)\quad (\mathbf{A}+\mathbf{B})*\mathbf{C}=\mathbf{A}*\mathbf{C}+\mathbf{B}*\mathbf{C}$

$$(\mathbf{A}_1\cdot\mathbf{B}_1\cdot\mathbf{C}_1)*(\mathbf{A}_2\cdot\mathbf{B}_2\cdot\mathbf{C}_2)=(\mathbf{A}_1\cdot\mathbf{A}_2)^T\cdot(\mathbf{B}_1\otimes\mathbf{B}_2)\cdot(\mathbf{C}_1\mathbf{C}_2)$$

4 Based on *quadratic*, *non–singular* matrices $\mathbf{A},\ \mathbf{B},\ \mathbf{C}\in\mathbb{R}^{n\times n}$, $\dim\mathbf{A}=\dim\mathbf{B}=\dim\mathbf{C}=n\times n$ the following *division-algebra* with respect to the *Cayley-product* is set-up.

$$\mathbf{A}=[a_{ij}],\ \mathbf{B}=[b_{ij}],\ \mathbf{C}=[c_{ij}]$$

$$(G1\cdot)\quad (\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}=\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$$

$$(G2\cdot)\quad \mathbf{A}\cdot\mathbf{I}=\mathbf{A}$$

$$(G3\cdot)\quad \mathbf{A}\cdot\mathbf{A}^{-1}=\mathbf{I}$$

The non-singular matrix $\mathbf{A}^{-1}=\mathbf{B}$, $\dim\mathbf{B}=n\times n$ the inverse matrix of $\mathbf{A}$ also called the *Cayley-inverse*, fulfils both equivalent conditions

$$\mathbf{A}\cdot\mathbf{B}=\mathbf{I}_n\Leftrightarrow\mathbf{B}\cdot\mathbf{A}=\mathbf{I}_n\ .$$

The *Cayley-inverse* is left-and right-identical. A *constructive representation* of the *Cayley-inverse* is

$$\mathbf{A}^{-1}=\frac{adj\mathbf{A}}{\det\mathbf{A}}$$

with respect to the adjoint matrix $adj\mathbf{A}$. adj A is generated as following: When from **A** the elements of its ith row and jth column are removed, the determinant of the remaining (n-1)-quadratic matrix is called a first minor of **A** and denoted by $|\mathbf{M}_{ij}|$. The *signed minor* $(-1)^{i+j}|\mathbf{M}|=:\alpha_{ij}$ is called the cofactor of a_{ij}. Then by definition $adj\mathbf{A}=[\alpha_{ij}]^T$. A numerical example is

$$\mathbf{A}=\begin{bmatrix}1&2&3\\2&3&2\\3&3&4\end{bmatrix}\in\mathbf{Z}^{3\times3},\quad adj\mathbf{A}=\begin{bmatrix}\alpha_{11}&\alpha_{21}&\alpha_{31}\\\alpha_{12}&\alpha_{22}&\alpha_{32}\\\alpha_{13}&\alpha_{23}&\alpha_{33}\end{bmatrix}$$

$\alpha_{11}=6$, $\alpha_{12}=-2$, $\alpha_{13}=-3$, $\alpha_{21}=1$, $\alpha_{22}=-5$, $\alpha_{23}=3$, $\alpha_{31}=-5$, $\alpha_{32}=4$, $\alpha_{33}=-1$

$$adj\mathbf{A}=\left[\alpha_{ij}\right]^T=\begin{bmatrix}6&1&-5\\-2&-5&4\\-3&3&-1\end{bmatrix},\det\mathbf{A}=-7$$

$$\mathbf{A}^{-1}=adj\mathbf{A}/\det\mathbf{A}=-\frac{1}{7}\begin{bmatrix}6&1&-5\\-2&-5&4\\-3&3&-1\end{bmatrix}$$

2-9 Complex algebra as a division algebra as well as a composition algebra, Clifford algebra $C\ell(0,1)$

Example 2-4: Complex algebra as a division algebra as well as a composition algebra, Clifford algebra, $C\ell(0,1)$

The “*complex algebra*” $\mathbb{C}$ due to C. F. Gauss is a division algebra over $\mathbb{R}$ as well as a composition algebra.

$$\mathbf{x}\in\mathbb{C}$$

$$\mathbf{x}=e_0\mathbf{x}^0+e_1\mathbf{x}^1 \text{ subject to } \left\{x^0,x^1\right\}\in\mathbb{R}^2$$

$$span\mathbb{C}=\left\{1,\mathbf{e}_1\right\}$$

[1] $\mathbf{x},\ \mathbf{y},\ \mathbf{z}\in\mathbb{R}^2$

$$\alpha(\mathbf{x},\mathbf{y})=:\mathbf{x}+\mathbf{y}$$

The axioms (G1+), (G2+), (G3+), (G4+) of an *Abelian group* apply.

[2] $\mathbf{x},\ \mathbf{y},\ \mathbf{z}\in\mathbb{R}^2,\quad r,s\in\mathbb{R}$

$$\beta(r,\mathbf{x})=:r\times\mathbf{x}\ .$$

The axioms (D1+), (D2+), (D3) of *additive distributivity* apply.

[3] One way to explicitly describe a multiplicative group with finitely many elements is to give a table listing the multiplications just representing the map $\gamma:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$

multiplication diagram, Cayley diagram

	1	$\mathbf{e}_1$
1	1	$\mathbf{e}_1$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1

Note that in the multiplication table each entry of a group appears exactly once in each row and column. The multiplication has to be read from the *left to right* that is the entry at the intersection of the row headed by $\mathbf{e}_1$ and the column headed by $\mathbf{e}_1$ is the product $\mathbf{e}_1 * \mathbf{e}_1$. Such a table is called a *Cayley diagram* of the multiplicative group. Here note in addition the associativity of the internal multiplication given in the table. Such a "*complex algebra*" $\mathbb{C}$ is *not a lie algebra* since neither $\mathbf{x} * \mathbf{x} = 0$ (**L**1) *nor* $(\mathbf{x} * \mathbf{y}) * \mathbf{z} + (\mathbf{y} * \mathbf{z}) * \mathbf{x} + (\mathbf{z} * \mathbf{x}) * \mathbf{y} = 0$ (Jacobi identity) (**L**2) hold. Just by means of the *multiplication table compute*

$$\mathbf{x} * \mathbf{x} = 1\{(\mathbf{x}^0)^2 - (\mathbf{x}^1)^2\} + 2\mathbf{e}_1\mathbf{x}^0\mathbf{x}^1 \neq 0 .$$

4 Begin with the choice

$$\mathbf{x}^{-1} = \frac{1}{(\mathbf{x}^0)^2 + (\mathbf{x}^1)^2}(\mathbf{1x}^0 - \mathbf{e}_1\mathbf{x}^1)$$

in order to end up with

$$\mathbf{x} * \mathbf{x}^{-1} = (\mathbf{1x}^0 + \mathbf{e}_1\mathbf{x}^1) * \frac{(\mathbf{1x}^0 - \mathbf{e}_1\mathbf{x}^1)}{(\mathbf{x}^0)^2 + (\mathbf{x}^1)^2} = 1$$

accordingly (G1∗), (G2∗), (G3∗) of a *division algebra* apply.

5 Begin with the choice

$$Q(\mathbf{x}) = Q(1\mathbf{x}^0 + \mathbf{e}_1\mathbf{x}^1) := (\mathbf{x}^0)^2 + (\mathbf{x}^1)^2$$

In order to prove (K1), (K2) and (K3). We only focus on (K2i):

$$Q(\mathbf{x} * \mathbf{y}) = Q(\mathbf{x}) \times Q(\mathbf{y}) \quad \text{(multiplicativity)}$$

$$Q(\mathbf{x} * \mathbf{y}) = Q\left\{1\left(x^0y^0 - x^1y^1\right) + e_1(x^1y^0 + x^0y^1)\right\} =$$

$$= (x^0)^2(y^0)^2 + (x^1)^2(y^1)^2 + (x^1)^2(y^0)^2 + (x^0)^2(y^1)^2$$

$$Q(\mathbf{x}) \times Q(\mathbf{y}) = \left\{(x^0)^2 + (x^1)^2\right\} \times \left\{(y^0)^2 + (y^1)^2\right\} =$$

$$= (x^0)^2(y^0)^2 + (x^1)^2(y^1)^2 + (x^1)^2(y^0)^2 + (x^0)^2(y^1)^2$$

$$\boxed{Q(\mathbf{x} * \mathbf{y}) = Q(\mathbf{x}) \times Q(\mathbf{y}) \quad \text{(q.e.d.)}}$$

How can be dream about such a complex algebra $\mathbb{C}$? *C. F. Gauss* (Theoria residorum biquadraticum, commentatio secunda, Göttingische gelehrte Anzeigen 1831, Werke vol. II (pages 169-178, Göttingen (1887) had been motivated in his number theory to introduce *complex numbers* with $i := \sqrt{-1}$ as the "*imaginary unit*". Identify $\mathbf{1x}^0$ with the "real part" and $\mathbf{e}_1\mathbf{x}^1 = i\mathbf{x}^1$ with the "imaginary part" of $\mathbf{x}$ and we are left with the standard theory of *complex numbers*. $\mathbf{x}^{-1}$ is based upon the *complex conjugate* $\mathbf{1x}^0 - \mathbf{e}_1\mathbf{x}^1$ of $\mathbf{x}$ being divided by the norm of $\mathbf{x}$. There is a remarkable *isomorphism* between *complex numbers* and *complex algebra*. The *proper algebraic interpretation of complex numbers* is in terms of Clifford algebra $C\ell(0,1)$. Observe $g(\mathbf{e}_1,\mathbf{e}_1) = -1$ which interprets the *binary operation* of the base vector which spans the *vector part* of a complex number. Now translate the multiplication table into the language of the Clifford product namely

$$1 \overset{*}{\wedge} 1 = 1, \quad 1 \overset{*}{\wedge} \mathbf{e}_1 = \mathbf{e}_1$$
$$\mathbf{e}_1 \overset{*}{\wedge} 1 = \mathbf{e}_1, \quad \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_1 = -1$$

in order to convince yourself that the Clifford algebra $C\ell(0,1)$ is algebraically isomorphic to the space of *complex numbers*. ♣

How can we relate *complex numbers* to *Clifford algebra* $C\ell_1$? Observe $g(\mathbf{e}_1,\mathbf{e}_1) = -1$ which interprets the *binary operation* of the base vector which spans the *vector part* of a complex number. While the *scalar part* of complex number is an element of $\mathbf{A}^0$, its *vector part* can be considered to be an element of $\mathbf{A}^1$. The *direct sum*

$$\mathbf{A}^0 \oplus \mathbf{A}^1$$

of spaces $\mathbf{A}^0$, $\mathbf{A}^1$ is algebraically isomorphic to the space of *complex numbers*, being an element of the *Clifford algebra* $C\ell_1$. ♣

2-10 Quaternion algebra as a division algebra as well as composition algebra, *Clifford algebra* $C\ell(0,2)$

Example 2-5: Quaternion algebra as a division algebra as well as composition algebra, *Clifford algebra* $C\ell(0,2)$

The "quaternion algebra" $\mathbb{H}$ due to W. R. Hamilton (1843) is a division algebra over $\mathbb{R}$ as well as a composition algebra:

$$\mathbf{x} \in \mathbb{H}$$

$$\mathbf{x} = \mathbf{1}x^0 + \mathbf{e}_1 x^1 + \mathbf{e}_2 x^2 + \mathbf{e}_3 x^3 \quad \textit{subject to} \{x^0, x^1, x^2, x^3\} \in \mathbb{R}^4$$

$$\operatorname{span}\mathbb{H} = \{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

1

$$\mathbf{x},\ \mathbf{y},\ \mathbf{z} \in \mathbb{R}^4$$

$$\alpha(\mathbf{x},\ \mathbf{y}) =: \mathbf{x} + \mathbf{y}\ .$$

The axioms (G1+), (G2+), (G3+), (G4+) of an *Abelian additive group* apply.

2

$$\mathbf{x},\ \mathbf{y} \in \mathbb{R}^4,\ r, s \in \mathbb{R}$$

$$\beta(r, \mathbf{y}) =: r \times \mathbf{x}\ .$$

The axioms (D1+), (D2+), (D3) of *additive distributivity* apply.

3 One way to explicitly describe a *multiplicative group* with finitely many elements is to give a *table listing the multiplications* just representing the map $\gamma : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$.

multiplication table, Cayley diagram

	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
1	**1**	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_1$	$\mathbf{e}_1$	**-1**	$\mathbf{e}_3$	$-\mathbf{e}_2$
$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{e}_3$	**-1**	e_1
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	**-1 .**

Note that in the multiplication table each entry of a group appears exactly once in each row and column. He multiplication has to be read *from left to right* that is, the entry at the intersection of the row headed by $\mathbf{e}_1$ and the column headed by $\mathbf{e}_2$ is the product $\mathbf{e}_1 * \mathbf{e}_2$. Such a table is called a *Cayley diagram* of the multiplicative group.

Here note in addition *associativity* of the internal multiplication given by the table, e. g. $\mathbf{e}_1 * (\mathbf{e}_2 * \mathbf{e}_3) = \mathbf{e}_1 * \mathbf{e}_1 = -1 = (\mathbf{e}_3 * \mathbf{e}_3) * \mathbf{e}_3$ or

$$\mathbf{x} * \mathbf{y} = 1(x^0 y^0 - \sum_{k=1}^{3} x^k y^k) + \sum_{i,j,k} \mathbf{e}_k (x^0 y^k + x^k y^0 + \varepsilon_{ij}^k x^i y^j)$$

such that $(\mathbf{x} * \mathbf{y}) * \mathbf{z} = \mathbf{x} * (\mathbf{y} * \mathbf{z})$. Such a "*Hamilton algebra*" $\mathbb{H}$ is *not* a *lie algebra* since *neither* $\mathbf{x} * \mathbf{x} = 0$ (**L**1) *nor* $(\mathbf{x} * \mathbf{y}) * \mathbf{z} + (\mathbf{y} * \mathbf{z}) * \mathbf{x} + (\mathbf{z} * \mathbf{x}) * \mathbf{y}) = 0$ (**L**2) (*Jacobi identity*) hold. Just by means of the *multiplication table* compute

$$\mathbf{x} * \mathbf{x} = 1\{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2\} + 2\mathbf{e}_1 x^0 x^1 + 2\mathbf{e}_2 x^0 x^2 + 2\mathbf{e}_3 x^0 x^3 \neq 0$$

4 Begin with the choice

$$\mathbf{x}^{-1} = \frac{1}{(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2} (1x^0 - \mathbf{e}_1 x^1 - \mathbf{e}_2 x^2 - \mathbf{e}_3 x^3)$$

in order *to end up* with

$$\mathbf{x} * \mathbf{x}^{-1} = (1x^0 + \mathbf{e}_1 x^1 + \mathbf{e}_2 x^2 + \mathbf{e}_3 x^3) * \frac{1x^0 - \mathbf{e}_1 x^1 - \mathbf{e}_2 x^2 + \mathbf{e}_3 x^3}{(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2} = 1 .$$

Accordingly (G1*),(G2*), (G3*) of a division algebra apply.

5 Begin with the choice

$$Q(\mathbf{x}) = Q(1x^0 + \mathbf{e}_1 x^1 + \mathbf{e}_2 x^2 + \mathbf{e}_3 x^3) := (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

in order to prove (K1), (K2) and (K3).

The laborious proofs are left as an *exercise*.

How can one dream about such a "quaternion algebra" $\mathbb{H}$? W. R. Hamilton (16 Oct 1843) invented *quaternion numbers* as outlined in a letter (1865) to his sun A. H. Hamilton for the following reason:

> "If I may be allowed to speak of *myself* in connection with the subject, I might do so in away with would bring you in, by referring to an *antiquaternionic* time, when you were a mere *child*, but had caught from me the conception of a Vector, as represented by a *Triplet;* and indeed I happen to be able to put the finger of memory upon the year an month – October, 1843 – when having recently returned from visits to Corp and Parsonstown, connected with a Meeting of the British Association, the desire to discover the laws of the multiplication referred to regained with me a certain strength and earnestness, with had for years been dormant, but was then on the point of being gratified, and was occasionally talked of with you. Every morning in the early part of the above cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me, "well, Papa, can you multiply triplets"? Were to a was always obliged to reply, with a sad shake of the head: "no, I can only add and subtract then".
>
> But on the 16^{th} day of the same month –with happened to be a Monday, and a Council day of the Royal Irish Academy – I was walking in to attend and preside, and your mother was working with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet an *under-current* of thought was going in my mind, which gave at last a result, were of it is not to much to say that I felt at once the importance. An electric circuit seemed to closed; and the spark flashed fort. The herald (as I fore saw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all evens on the parts of others, if should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse – as philosophical as it may have been – to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, **i**, **j**, **k** namely
>
> $$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$
>
> which contains the Solution of the problem, but of course, as an inscription, has long since moldered away. A more durable notice remains, however, on the Council Books of the Academy for that day (Oct 16^{th}, 1843), which records the fact, that I then asked for and obtained based to read a Paper on Quaternion, ad the First General Meeting of the Session: which reading took place accordingly, on Monday the 13^{th} of the November following."

Obviously the *vector part* $\mathbf{e}_1 x^1 + \mathbf{e}_2 x^2 + \mathbf{e}_3 x^3 = \mathbf{i}x^1 + \mathbf{j}x^2 + \mathbf{k}x^3$ of a quaternion number replaces the *imaginary part* of a parted of a complex number, the *scalar part* $1\mathrm{x}^0$ the *real part*. The *quaternion conjugate*

$$1x^\circ - \sum_{k=1}^{3} \mathbf{e}_k x^k =: \mathbf{x}^*$$

Substitutes the *complex conjugate* of the complex number, leading to the quaternion inverse

$$\mathbf{x}^{-1} = \frac{\mathbf{x}^*}{Q(\mathbf{x})}.$$

The proper algebraic interpretation of *quaternion numbers* is in terms of *Clifford algebra* $C\ell(0,2)$. If $n = \dim \mathbb{X} = 2$ is the dimension of the *linear space* $\mathbb{X}$ which we base the *Clifford algebra* on, its bases elements are

$$\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2\}$$

subject to

$$\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 + \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_1 = 0,$$

$$\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_1 = g(\mathbf{e}_1, \mathbf{e}_2)1 = \text{-}1, \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_2 = g(\mathbf{e}_2, \mathbf{e}_2) = \text{-}1,$$

$$(\mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_2)^2 = (\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2) \overset{*}{\wedge} (\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2) = \text{-}\mathbf{e}_2 \overset{*}{\wedge} (\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_1) \overset{*}{\wedge} \mathbf{e}_1 = +\mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_2 = \text{-}1$$

$$(\mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_2) \overset{*}{\wedge} \mathbf{e}_1 = \text{-}\mathbf{e}_2 \overset{*}{\wedge} (\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_1) = \mathbf{e}_2$$

$$(\mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_2) \wedge \mathbf{e}_2 = \mathbf{e}_1 \overset{*}{\wedge} (\mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_2) = \text{-}\mathbf{e}_1$$

and

$$\mathbf{e}_3 := \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2$$

by *classical notation*. Obviously

$$\mathbf{x} = 1x^0 + \mathbf{e}_1 x^1 + \mathbf{e}_2 x^2 + \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 x^3 \in C\ell(0,2)$$

is an element of *Clifford algebra* $C\ell(0,2)$.

There is an algebraic isomorphism between the *quaternion algebra* $\mathbb{H}$ *of vectors* and the *quaternion algebra of the matrices*, namely either $\mathbb{M}(\mathbb{R}^{4\times 4})$ of 4×4 *real matrices* or $\mathbb{M}(\mathbb{C}^{2\times 2})$ of 2×2 *complex matrices.*

Firstly we define the 4×4 real matrix basis $\mathbf{E}$ and decompose it into the four constituents $\mathbf{\Sigma}_0, \mathbf{\Sigma}_1, \mathbf{\Sigma}_2, \mathbf{\Sigma}_3$ of 4×4 of *real Pauli matrices* which form a multiplicative group of the multiplication table of Hamilton type

$$\mathbf{E} = \begin{bmatrix} 1 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -\mathbf{e}_1 & 1 & -\mathbf{e}_3 & \mathbf{e}_2 \\ -\mathbf{e}_2 & \mathbf{e}_3 & 1 & -\mathbf{e}_1 \\ -\mathbf{e}_3 & -\mathbf{e}_2 & \mathbf{e}_1 & 1 \end{bmatrix} = 1\mathbf{\Sigma}_0 + \mathbf{e}_1\mathbf{\Sigma}_1 + \mathbf{e}_2\mathbf{\Sigma}_2 + \mathbf{e}_3\mathbf{\Sigma}_3 \in \mathbb{R}^{4\times4}$$

$$\mathbf{\Sigma}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{\Sigma}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{\Sigma}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \mathbf{\Sigma}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

multiplication table, Cayley diagram

	$\mathbf{\Sigma}_0$	$\mathbf{\Sigma}_1$	$\mathbf{\Sigma}_2$	$\mathbf{\Sigma}_3$
$\mathbf{\Sigma}_0$	$\mathbf{\Sigma}_0$	$\mathbf{\Sigma}_1$	$\mathbf{\Sigma}_2$	$\mathbf{\Sigma}_3$
$\mathbf{\Sigma}_1$	$\mathbf{\Sigma}_1$	$-\mathbf{\Sigma}_0$	$\mathbf{\Sigma}_3$	$-\mathbf{\Sigma}_2$
$\mathbf{\Sigma}_2$	$\mathbf{\Sigma}_2$	$-\mathbf{\Sigma}_3$	$-\mathbf{\Sigma}_0$	$\mathbf{\Sigma}_1$
$\mathbf{\Sigma}_3$	$\mathbf{\Sigma}_3$	$\mathbf{\Sigma}_2$	$-\mathbf{\Sigma}_1$	$-\mathbf{\Sigma}_0$

or

$$\mathbf{\Sigma}_0\mathbf{\Sigma}_0 = \mathbf{\Sigma}_0, \quad \mathbf{\Sigma}_0\mathbf{\Sigma}_i = \mathbf{\Sigma}_i\mathbf{\Sigma}_0 = \mathbf{\Sigma}_i \qquad \text{for all } i \in \{1,2,3\}$$

$$\mathbf{\Sigma}_i\mathbf{\Sigma}_j = -\delta_{ij}\mathbf{\Sigma}_0 + \varepsilon_{ijk}\mathbf{\Sigma}_k \qquad \text{for all } i, j, k \in \{1,2,3\}$$

Let $\mathbf{A},\mathbf{B},\mathbf{C} \in \mathbb{M}\left(\mathbb{R}^{4\times4}, \textit{Hamilton}\right)$ continued by means of

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_2 & a_1 & -a_4 & a_3 \\ -a_3 & a_4 & a_1 & -a_2 \\ -a_4 & -a_3 & a_2 & a_1 \end{bmatrix} =: \mathbf{A}$$

$$\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ -b_2 & b_1 & -b_4 & b_3 \\ -b_3 & b_4 & b_1 & -b_2 \\ -b_4 & -b_3 & b_2 & b_1 \end{bmatrix} =: \mathbf{B}$$

such that the *Cayley-product*

$$\mathbf{AB} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ -c_2 & c_1 & -c_4 & c_3 \\ -c_3 & c_4 & c_1 & -c_2 \\ -c_4 & -c_3 & c_2 & c_1 \end{bmatrix} =: \mathbf{C} \in \mathbb{M}\left(\mathbb{R}^{4\times 4}, Hamilton\right)$$

$$c_1 = a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4$$
$$c_2 = a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3$$
$$c_3 = a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2$$
$$c_4 = a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1$$

fulfilling the axioms $(G1\circ),(G2\circ),(G3\circ)$ of a *non-Abelian multiplicative group*, namely

$(G1\circ)$ $(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$ (*associativity*)

$(G2\circ)$ $\mathbf{AI}=\mathbf{A}$ (*identity*)

$(G3\circ)$ $\mathbf{AA}^{-1}=\mathbf{I}$ (*inverse*),

but $(G4\circ)$ does *not* apply, in particular

$$\det \mathbf{A} = a_1^2 + a_2^2 + a_3^2 + a_4^2, \quad \det \mathbf{B} = b_1^2 + b_2^2 + b_3^2 + b_4^2$$
$$(\det \mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}).$$

Secondly, we define the 2×2 complex matrix basis $\mathbf{E}$ and decompose it into the four constituents $\mathbf{\Sigma}^0, \mathbf{\Sigma}^1, \mathbf{\Sigma}^2, \mathbf{\Sigma}^3$ of 2×2 *complex Pauli matrices* which form a multiplicative group of the multiplication table of Hamilton type

$$\mathbf{E} := \begin{bmatrix} 1 + i\mathbf{e}_1 & \mathbf{e}_2 + i\mathbf{e}_3 \\ -\mathbf{e}_2 + i\mathbf{e}_3 & 1 - i\mathbf{e}_1 \end{bmatrix} = 1\mathbf{\Sigma}^0 + \mathbf{e}_1\mathbf{\Sigma}^1 + \mathbf{e}_2\mathbf{\Sigma}^2 + \mathbf{e}_3\mathbf{\Sigma}^3 \in \mathbb{C}^{2\times 2}$$

$$\mathbf{\Sigma}^0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{\Sigma}^1 := \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

$$\mathbf{\Sigma}^2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{\Sigma}^3 := \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

multiplication table, Cayley diagram

	$\mathbf{\Sigma}^0$	$\mathbf{\Sigma}^1$	$\mathbf{\Sigma}^2$	$\mathbf{\Sigma}^3$
$\mathbf{\Sigma}^0$	$\mathbf{\Sigma}^0$	$\mathbf{\Sigma}^1$	$\mathbf{\Sigma}^2$	$\mathbf{\Sigma}^3$
$\mathbf{\Sigma}^1$	$\mathbf{\Sigma}^1$	$-\mathbf{\Sigma}^0$	$\mathbf{\Sigma}^3$	$-\mathbf{\Sigma}^2$
$\mathbf{\Sigma}^2$	$\mathbf{\Sigma}^2$	$-\mathbf{\Sigma}^3$	$-\mathbf{\Sigma}^0$	$\mathbf{\Sigma}^1$
$\mathbf{\Sigma}^3$	$\mathbf{\Sigma}^3$	$\mathbf{\Sigma}^2$	$-\mathbf{\Sigma}^1$	$\mathbf{\Sigma}^0$

Note that $E \in \mathbb{C}^{2\times 2}$ is "*Hermitean*". Let $\mathbf{A},\mathbf{B},\mathbf{C} \in \mathbb{M}$ $(\mathbb{C}^{2\times 2}, Hamilton)$ constituted by means of

$$\begin{bmatrix} a_1 + ia_2 & a_3 + ia_4 \\ -a_3 + ia_4 & a_1 - ia_2 \end{bmatrix} := \mathbf{A}$$

$$\begin{bmatrix} b_1 + ib_2 & b_3 + ib_4 \\ -b_3 + ib_4 & b1 - ib_2 \end{bmatrix} := \mathbf{B}$$

such that the *Cayley-product*

$$\mathbf{AB} = \begin{bmatrix} c_1 + ic_2 & c_3 + ic_4 \\ -c_3 + ic_4 & c_1 - ic_2 \end{bmatrix} =: \mathbf{C} \in \mathbb{M} \ (\mathbb{C}^{2\times 2}, \ \mathit{Hamilton})$$

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) =$$

$$= (a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) = c_1^2 + c_2^2 + c_3^2 + c_4^2$$

$$c_1 + ic_2 = (a_1 + ia_2)(b_1 + ib_2) + (a_3 + ia_4)(-b_3 + ib_4) =$$

$$= a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4 + i(a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3)$$

$$c_3 + ic_4 = (a_1 + ia_2)(b_3 + ib_4) + (a_3 + ia_4)(b_1 - ib_2) =$$

$$= a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2 + i(a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1)$$

fulfilling the axioms $(G1\circ), (G2\circ), (G3\circ)$ of a *non-Abelian multiplicative group*. The *spinor*

$$s := \begin{bmatrix} 1 + i\mathbf{e}_1 \\ \mathbf{e}_2 + i\mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

as a vector of length zero relates to the 2×2 *complex matrix basis* by

$$\mathbf{E} = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix}. \qquad ♣$$

Example 2-6: Octonian algebra as a non-associative algebra as well as a composition algebra, *Clifford algebra with respect to* $\mathbb{H} \times \mathbb{H}$

The *octonian algebra* $\mathbb{O}$ also called "*the algebra of octaves*" due to *J. T. Graves* (1843) and *A. Cayley* (1845) is a *composition algebra over* $\mathbb{R}$ as well as a *non-associative algebra:*

$$\mathbf{x} \in \mathbb{O}$$

$$\mathbf{x} = 1x^0 + \mathbf{e}_1 x^1 + \mathbf{e}_2 x^2 + \mathbf{e}_3 x^3 + \mathbf{e}_4 x^4 + \mathbf{e}_5 x^5 + \mathbf{e}_6 x^6 + \mathbf{e}_7 x^7$$

subject to

$$\{x^1, x^2, ..., x^6, x^7\} \in \mathbb{R}^8$$

$$\mathit{span}\mathbb{O} = \{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$$

1

$$\mathbf{x}, \ \mathbf{y}, \ \mathbf{z} \in \mathbb{R}^8$$

$$\alpha(\mathbf{x}, \mathbf{y}) =: \mathbf{x} + \mathbf{y}$$

The axioms $(G1+),(G2+),(G3+),(G4+)$ of an *Abelian additive group* apply.

2
$$\mathbf{x},\mathbf{y}\in\mathbb{R}^8,\quad r,s\in\mathbb{R}$$
$$\beta(r,\mathbf{x}) =: r\times\mathbf{x}$$

The axioms $(D1+),(D2+),(D3)$ of *additive distributivity* apply.

3 One way to explicitly describe a *multiplicative group* with finitely many elements is to give a *table* listing the *multiplications* just representing the map $\gamma:\mathbb{O}\times\mathbb{O}\to\mathbb{O}$.

multiplication table, Cayley diagram

	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_3$	-1	$\mathbf{e}_1$	$-\mathbf{e}_6$	$\mathbf{e}_7$	$-\mathbf{e}_4$	$-\mathbf{e}_5$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	-1	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$
$\mathbf{e}_4$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	-1	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$
$\mathbf{e}_5$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_1$	-1	$-\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{e}_6$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_2$	$\mathbf{e}_3$	-1	$-\mathbf{e}_1$
$\mathbf{e}_7$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

Note that in the multiplication table each entry of a group appears exactly once in each row and column. The multiplication has to be read *from left to right* that is, the entry at the intersection of the row headed by $\boldsymbol{e}_5$ is the product $\mathbf{e}_3*\mathbf{e}_5$. Such a table is called a *Cayley diagram* of the multiplicative group.
Note the non-associativity of the internal multiplication given by the table, e.g. $\mathbf{e}_2*(\mathbf{e}_3*\mathbf{e}_4)\neq(\mathbf{e}_2*\mathbf{e}_3)*\mathbf{e}_4$, namely by means of $\mathbf{e}_3*\mathbf{e}_4 =$ $\mathbf{e}_7$, $\mathbf{e}_2*(\mathbf{e}_3*\mathbf{e}_4)=\mathbf{e}_2*\mathbf{e}_7=-\mathbf{e}_5$ versus $\mathbf{e}_2*\mathbf{e}_3=\mathbf{e}_1$, $(\mathbf{e}_2*\mathbf{e}_3)*\mathbf{e}_4$ $=\mathbf{e}_1*\mathbf{e}_4=+\mathbf{e}_5$. Such an "octonian algebra" $\mathbb{O}$ is not a Lie algebra since neither $\mathbf{x}*\mathbf{x}=0$ $(\mathbf{L}1)$ nor $(\mathbf{x}*\mathbf{y})*\mathbf{z}+(\mathbf{y}*\mathbf{z})*\mathbf{x}+(\mathbf{z}*\mathbf{x})*\mathbf{y}=0$ $(\mathbf{L}2)$ *(Jacobi identity)* hold. Just by means of the multiplication table compute
$$(\mathbf{e}_2*\mathbf{e}_3)*\mathbf{e}_4+(\mathbf{e}_3*\mathbf{e}_4)*\mathbf{e}_2+(\mathbf{e}_4*\mathbf{e}_2)*\mathbf{e}_3=\mathbf{e}_5\neq 0$$

4 does not apply.

5 Begin with the choice
$$Q(\mathbf{x})=Q(1x^0+e_1x^1+\ldots+e_6x^6+e_7x^7)=(x^0)^2+(x^1)^2+\ldots+(x^6)^2+(x^7)^2$$
in order to prove (K1), (K2), (K3). The laborious proofs are left as an exercise.

The proper algebraic interpretation of octonian numbers is in terms of *Clifford algebra*, namely with respect to the *eight dimensional* set $\mathbb{H}\times\mathbb{H}=:\mathbb{H}^2$

where $\mathbb{H}$ is the usual skew field of Hamilton's quaternions, algebraically isomorphic to $C\ell(0,2)$. Indeed it would have temptation to base "octonian algebra" $\mathbb{O}$ on $C\ell(0,3)$, $\dim C\ell(0,3) = 2^3 = 8$, but its *generic elements*

$$\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2, \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_3, \mathbf{e}_3 \overset{*}{\wedge} \mathbf{e}_1, \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_3\}$$

are not representing the *octonian multiplication table* e.g.

$$(\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_3)^2 = g(\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_3, \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_3) =$$
$$= (\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_3) \overset{*}{\wedge} (\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_3) =$$
$$= -\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_3 \overset{*}{\wedge} \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_3 = \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 \overset{*}{\wedge} (\mathbf{e}_3 \overset{*}{\wedge} \mathbf{e}_3) =$$
$$= -\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 = -\mathbf{e}_1 \overset{*}{\wedge} (\mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_2) \overset{*}{\wedge} \mathbf{e}_1 =$$
$$= -(\mathbf{e}_2 \overset{*}{\wedge} \mathbf{e}_1) = +1$$

In contrast, let us introduce the *pair*

$$\mathbf{x} := (\mathbf{a}, \mathbf{b}) \in \{\mathbb{X} \mid \mathbf{a} \in \mathbb{H},\ \mathbf{b} \in \mathbb{H}\}$$
$$\mathbf{x} \in \mathbb{H}^2,\ \mathbf{x}' \in \mathbb{H}^2,\ \mathbf{x}'' \in \mathbb{H}^2$$

1
$$\alpha(\mathbf{x}, \mathbf{x}') =: \mathbf{x} + \mathbf{x}'$$
$$\mathbf{x} + \mathbf{x}' = (\mathbf{a} + \mathbf{a}',\ \mathbf{b} + \mathbf{b}').$$

The axioms $(D1+), (D2+), (D3+), (D4+)$ of an *Abelian additive group* apply.

$$\mathbf{x}, \mathbf{x}' \in \mathbb{H}^2,\ \ r, r' \in \mathbb{R}$$

2
$$\beta(r, \mathbf{x}) =: r \times \mathbf{x}$$
$$r \times \mathbf{x} = (r \times \mathbf{a}, r \times \mathbf{b}).$$

The axioms $(D1+), (D2+), (D3)$ of *additive distributivity* apply.

$$\mathbf{x} \in \mathbb{H} \times \mathbb{H} = \mathbb{H}^2,\ \ \mathbf{x}' \in \mathbb{H} \times \mathbb{H} = \mathbb{H}^2$$

3
$$\gamma(\mathbf{x}, \mathbf{x}') =: \mathbf{x} * \mathbf{x}'$$
$$\mathbf{x} * \mathbf{x}' =: (\mathbf{a}\mathbf{a}' - \overline{\mathbf{b}}'\mathbf{b},\ \ \mathbf{b}'\mathbf{a} + \mathbf{b}\overline{\mathbf{a}}')$$

$\overline{\mathbf{a}}, \overline{\mathbf{b}}$ denote the conjugate of $\mathbf{a} \in \mathbb{H},\ \mathbf{b} \in \mathbb{H}$, respectively. If $(\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}')$ are represented by

$$(1\alpha^0 + \sum_{i=1}^{3} \mathbf{e}_i \alpha^i,\ 1\beta^0 + \sum_{j=1}^{3} \mathbf{e}_j \beta^j),\ (1\alpha'^{\circ} + \sum_{i'=1}^{3} \mathbf{e}_{i'} \alpha'^i,\ 1\beta'^{\circ} + \sum_{j'=1}^{3} \mathbf{e}_{j'} \beta'^j)$$

respectively, where

$$span\mathbb{H} = \{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_2 = \mathbf{e}_3\}$$

or

$$span\mathbb{H} = \{1', \mathbf{e}_{1'}, \mathbf{e}_{2'}, \mathbf{e}_{1'} \overset{*}{\wedge} \mathbf{e}_{2'} = \mathbf{e}_{3'}\} = \{1', \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$$

the *"octonian product"* $\mathbf{x}*\mathbf{x}'$ results in

$$\mathbf{x}*\mathbf{x}' =$$

$$(1[\alpha^0\alpha'^0 - \beta^0\beta'^0 - \sum_{k=1}^{3}(\alpha^k\alpha'^k - \beta^k\beta'^k)] +$$

$$+\sum_{i,j,k=1}^{3} e_k[\alpha^0\beta^k + \alpha^k\beta^0 + e_{ij}^k(\alpha^i\alpha'^j - \beta^j\beta'^i)],$$

$$1[\alpha^0\beta'^0 - \alpha'^0\beta^0 - \sum_{k=1}^{3}(\alpha^k\beta'^k - \alpha'^k\beta^k)] +$$

$$+\sum_{i,j,k=1}^{3} e_k[\alpha^0\beta'^k + \alpha^k\beta'^0 + e_{ij}^k(\alpha^j\beta'^i - \alpha'^j\beta^i)]).$$

the axioms $(D1*+), (D2*\times)$ of *distributivity* apply. The pair (1,0) is the neutral element.

4 *Begin* with the choice of

(1st) $\bar{\mathbf{x}} := (\bar{\mathbf{a}}, -\mathbf{b})$ of $\mathbf{x} \in \mathbb{H}^2$ - the transpose

(2nd) $\mathbf{x}*\bar{\mathbf{x}} = Q(\mathbf{x})$ or $\mathbf{x}*\bar{\mathbf{x}} = (\mathbf{a}\bar{\mathbf{a}} + \mathbf{b}\bar{\mathbf{b}}, \bar{0})$

(3rd) $\mathbf{x}^{-1} = \dfrac{\bar{\mathbf{x}}}{Q(\mathbf{x})}$ if $\mathbf{x} \neq 0$

in order to *end up* with

$$\mathbf{x}*\mathbf{x}^{-1} = 1\,.$$

A historical perspective of octonian numbers is given by *B. L. van Waerden*: Hamilton's discovery of quaternions, Mathematical Magazine *49* (1976) 227-234. Reference is made to *J. T. Graves*: Transactions of the Irish Academy *21* (1848) 338- and *A. Cayley*: Collected Mathematical Papers, vol. 1, page 127 and vol.11, pages 368-371. ♣

The exceptional role of the examples 1-10, 1-11 and 1-12 on *complex, quaternion and octonian algebra* illustrating Clifford algebra $C\ell(0,1)$, $C\ell(0,2)$ as well as Clifford algebra with respect to $\mathbb{H}^2$ is *established* by the following theorems:

Theorem 2-1 ("*Hurwitz' theorem of composition algebras* "):

A complete list of composition algebras over $\mathbb{R}$ *consists of*

(i) the real numbers $\mathbb{R}$,

(ii) the complex numbers $\mathbb{C}$,

(iii) the quaternions $\mathbb{H}$,

(iv) the octonians $\mathbb{O}$.

Theorem 2-2 ("*Frobenius' theorem of division algebras* "):

The only finite-dimensional division algebra over $\mathbb{R}$ *are*

(α) the real numbers $\mathbb{R}$,

(β) the complex numbers $\mathbb{C}$,

(γ) the quaternions $\mathbb{H}$.

Historical Aside

For details consult the historical texts *A. Hurwitz*: "Über die Composition der quadratischen Formen von beliebig vielen Variablen, Nachrichten Ges. Wiss. Göttingen (1898) 309-316, *G. Frobenius*: "Über lineare Substitutionen und bilineare Formen, Crelle's J. Reine angewandte Math. *84*" (1878) 1 - 63 as well as *U. Haslet*: on the theory of associative division algebras, Trans. American Math. Soc. *18* (1917) 167-176. A more recent reference is *N. Jacobson* (1974, pages 425 and 430).

Chapter 3
The algebra of antisymmetric and symmetric tensor-valued functions

While we already introduced the decompositions of multilinear functions into *symmetric, antisymmetric* and *residual* multilinear functions, we shall treat the algebra of antisymmetric and symmetric tensor-valued functions in more detail, here. The algebra of *antisymmetric multilinear functions*, also called *Grassmann algebra*, *exterior algebra*, is built on *(i)* the four axioms (G1+), (G2+), (G3+), (G4+), internal relations of type additions, *(ii)* the three axioms (D1x+), (D2+x), (D3+), external relations of type multiplication, namely distributivity, *and (iii)* the five axioms (G1∧), (D1∧+), (D2∧+), (D3∧x), (G4∧), internal relations of type exterior product, namely associativity, distributivity and anticommutativity. By means of *Corollary 3-1* we give the dimension of space of antisymmetric multilinear functions as well as the dimensions of the *direct sum* of the spaces of antisymmetric multilinear functions. *Corollary 3-2* states the *induced metric* of an antisymmetric multilinear function. *Example 3-1* is an extensive review of generating the *normal form* of an antisymmetric multilinear function, namely the decomposition into p-vectors, also called *product sum decompositions*. Alternatively the name "*blades*" is used. Finally the algebra of symmetric multilinear functions, also called *interior algebra*, is constructed by *(i)* the four axioms (G1+), (G2+), (G3+), (G4+), internal relations of type *addition*, *(ii)* the three axioms (D1+), (D2+), (D3+), external relations of type multiplication, namely *distributivity,* and *(iii)* the five axioms (G1∧), (D1∧+), (D2∧+), (D3∧x), (G4∧), internal relations of type interior product, namely *associativity, distributivity and commutativity*. The dimension of the space of symmetric multilinear functions as well as the dimension of the *direct sum* of the spaces of symmetric multilinear functions is summarized in *Corollary 3-3*.

3-1 Exterior Algebra, Grassmann Algebra

So prepared we shall structure *The algebra of antisymmetric and symmetric tensor-valued functions.* Let us begin with *exterior algebra*.

Definition 3-1: (*Grassmann algebra,* antisymmetric algebra, exterior algebra, algebra of antisymmetric multilinear functions):

In terms of a general coordinate base $\mathbb{X}^* = span\{\mathbf{b}^1,...\mathbf{b}^n\}$ let $\alpha \in \mathbf{A}^p, \beta \in \mathbf{A}^r, \gamma \in \mathbf{A}^t$ be antisymmetric multilinear functions on a linear space $\mathbb{X}$, namely

$$\alpha = \frac{1}{p!}\sum_{i_1,...,i_p=1}^{n=\dim \mathbb{X}^*} \mathbf{b}^{i_1} \wedge ... \wedge \mathbf{b}^{i_p} \alpha_{i_1,...,i_p} \in \mathbb{A}^p(\mathbb{X}^*)$$

$$\beta = \frac{1}{\mathrm{r}!}\sum_{j_1,...,j_r=1}^{n=\dim \mathbb{X}^*} \mathbf{b}^{j_1} \wedge ... \wedge \mathbf{b}^{j_r} \beta_{j_1,...,j_r} \in \mathbb{A}^b(\mathbb{X}^*)$$

$$\gamma = \frac{1}{\mathrm{t}!}\sum_{k_1,...,k_t=1}^{n=\dim \mathbb{X}^*} \mathbf{b}^{k_1} \wedge ... \wedge \mathbf{b}^{k_t} \gamma_{k_1,...,k_t} \in \mathbb{A}^t(\mathbb{X}^*)$$

An antisymmetric multilinear algebra over $\mathbb{R}$ (also called Grassmann algebra or exterior algebra) as a graded $\mathbb{R}$-algebra consists of an $\mathbf{A}^{\mathrm{p}}$, two internal relations (addition and inner multiplication) $+: \mathbf{A}^p \times \mathbf{A}^p \to \mathbf{A}^p, \wedge: \mathbf{A}^p \times \mathbf{A}^p \to \mathbf{A}^{p+r}$ and one external relation (external multiplication) $\mathbf{x}: \mathbb{R} \times \mathbf{A}^p \to \mathbf{A}^p$ where the following properties hold.

first: addition

$\alpha, \beta, \gamma \in \mathbf{A}^p$, "+" (internal relation of type addition)

(G1+) $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ (*associativity of addition*)

(G2+) $\alpha + 0 = \alpha$ (*identity of addition*)

(G3+) $\alpha - \alpha = 0$ (*inverse of addition*)

(G4+) $\alpha+\beta = \beta+\alpha$ (*commutativity of addition*)

second: multiplication

$\alpha, \beta \in \mathbf{A}^p$, $r, s \in \mathbb{R}$, "$\times$" (external relation of type multiplication)

($D1\times+$) $r\times(\alpha+\beta) = r\times\alpha + r\times\beta =$

$= \alpha\times r + \beta\times r = (\alpha+\beta)\times r$ (1*st distributivity*)

($D2+\times$) $(r\times s)\times\alpha = r\times\alpha + s\times\alpha =$

$= \alpha\times r + \alpha\times s = \alpha\times(r+s)$ (2*nd distributivity*)

($D3$) $1\times\alpha = \alpha\times 1 = \alpha$

third: exterior product

$\alpha \in \mathbf{A}^p, \beta \in \mathbf{A}^r, \gamma \in \mathbf{A}^t$, "$\wedge$" (internal relation of type exterior product)

$$\alpha\wedge\beta = \begin{cases} 0 & if \quad p+r>0 \\ \dfrac{1}{p!r!}\displaystyle\sum_{i_1,..,i_p,i_{p+1},...,i_{p+r}}^{n=\dim\mathbb{X}^*} b^{i_1}\wedge ...\wedge b^{i_p}\wedge b^{i_{p+1}}\wedge ...\wedge b^{i_{p+r}}\alpha_{i_1...i_p}\beta_{i_{\beta+1}...i_\beta} & if \quad p+r\le n \end{cases}$$

($G1\wedge$) $(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$

(associativity of internal multiplication of type exterior product)

$(D1\wedge +)\quad \alpha\wedge(\beta+\gamma)=\alpha\wedge\beta+\alpha\wedge\beta \quad if \quad \alpha\in\mathbf{A}^p, \beta,\gamma\in\mathbf{A}^r$
(additive distribuity w.r.t. internal multiplication of type exterior product)

$(D2\wedge +)\quad (\alpha+\beta)\wedge\ \gamma=\alpha\wedge\gamma+\beta\wedge\gamma \quad$ if $\alpha,\beta\in\mathbf{A}^p,\ \gamma\in\mathbf{A}^{\mathrm{r}}$
(additive distribuity w. r. t. internal multiplication of type exterior product)

$(G3\wedge\times)\quad \mathrm{r}\times(\alpha\wedge\beta)=(\mathrm{r}\times\alpha)\wedge\beta$
(distributivy of internal multiplication of type exterior product and external multiplication)

$(G4\wedge)\quad \beta\wedge\alpha=(-1)^{pr}\alpha\wedge\beta$
(graded anticommutativity of exterior multiplication)

Corollary 3-1: $(\dim\mathbf{A}^p,\ \dim\oplus_{p=\rho}^{w}\mathbf{A}^p)$:

$$\dim\mathbf{A}^p=\dim\wedge^p\mathbb{X}^*=\binom{n}{p}$$

$$\dim\oplus_{p=0}^{n}\mathbf{A}^p=\dim\oplus_{p=0}^{n}\wedge^p\mathbb{X}^*=2^n \quad ♣$$

The elements of the space which is generated by the *direct sum* of the spaces of antisymmetric multilinear functions, namely

$$\{\mathbf{A}^0\oplus\mathbf{A}^1\oplus....\oplus\mathbf{A}^n\}$$

are {scalars, vectors/differential one forms, (2,0) antisymmetric tensors / differential two forms, (3,0) antisymmetric tensors/ differential three forms, ..., (*n*, 0) antisymmetric tensors/ differential *n*-form}.

$$1\cdot f_0+\frac{1}{1!}\sum_{i_1=1}^{n=\dim\mathbb{X}^*}\mathbf{e}^{i_1}f_{i_1}+\frac{1}{2!}\sum_{i_1,i_2=1}^{n=\dim\mathbb{X}^*}\mathbf{e}^{i_1}\wedge\mathbf{e}^{i_2}f_{i_1i_2}+$$
$$+\frac{1}{3!}\sum_{i_1,i_2,i_3=1}^{n=\dim\mathbb{X}^*}\mathbf{e}^{i_1}\wedge\mathbf{e}^{i_2}\wedge\mathbf{e}^{i_3}f_{i_1i_2i_3}+\cdots+\frac{1}{n!}\sum_{i_1,\cdots,i_n=1}^{n=\dim\mathbb{X}^*}\mathbf{e}^{i_1}\wedge\cdots\wedge\mathbf{e}^{i_n}f_{i_1\cdots i_n}$$

or

$$f_0\cdot 1+\frac{1}{1!}\sum_{i_1=1}^{n=\dim\mathbb{X}^*}f_{i_1}dx^{i_1}+\frac{1}{2!}\sum_{i_1,i_2=1}^{n=\dim\mathbb{X}^*}f_{i_1i_2}dx^{i_1}\wedge dx^{i_2}+$$
$$+\frac{1}{3!}\sum_{i_1,i_2,i_3=1}^{n=\dim\mathbb{X}^*}f_{i_1i_2i_3}dx^{i_1}\wedge dx^{i_2}\wedge dx^{i_3}+\cdots+\frac{1}{n!}\sum_{i_1,\cdots,i_n=1}^{n=\dim\mathbb{X}^*}f_{i_1\cdots i_n}dx^{i_1}\wedge\cdots\wedge dx^{i_n}$$

are examples of zero rank *Clifford numbers* (*W. K. Clifford*: Application of Grassmann's extensive algebra, American J. Math. 1 (1878) 350-358, in particular page 353). More details are given later under *Clifford algebra*. Here we extend *Grassmann algebra* by

Corollary 3-2 (induced metric of an antisymmetric multilinear function):

Let g be a metric on an n-dimensional Euclidean space $\mathbb{E}^n = \{\mathbb{R}^n, g_{ij}\}$, let $\alpha, \beta \in \mathbb{A}^p$ be (p, 0) tensor-valued functions,

$$\mu := \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n = \sqrt{g}\,\mathbf{b}^1 \wedge \cdots \wedge \mathbf{b}^n$$

its volume element with respect to an orthonormal base $\{\mathbf{e}^1, \cdots, \mathbf{e}^n\}$ or $\{\mathbf{b}^1, \cdots, \mathbf{b}^n\}$ of neither orthogonal, nor normalized type. Then there is an *induced metric* defined by

$$g(\alpha, \beta) = \frac{1}{p!} \sum_{i_1, \cdots, i_p, j_1, \cdots, j_p}^{n=\dim \mathbb{X}^*} g^{i_1 j_1} \cdots g^{i_p j_p} \alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_p}$$

such that

$$\alpha \wedge {*\beta} = g(\alpha, \beta)\mu$$

holds.

3-2 The normal form of an antisymmetric multilinear function, product sum decomposition

When we present as early as by *definition 1-1 the axioms of multilinear functions* which constitute *multilinear algebra* or *tensor algebra* $\mathbb{T}^p_g$ over the field of real numbers we did not specify the linear map $g : \mathbb{T}^p_g \to \mathbb{X}$, $\dim \mathbb{X} = n$, namely the *inverse of the map* $f : \mathbb{X} \to \mathbb{T}^p_g$. For instance, *given* $\mathbf{e}^1 \wedge \mathbf{e}^2 f_{12} \in \mathbb{A}^2$, $\dim \mathbb{A}^2 = 1$, $\dim \mathbb{X} = 2$, as an element of the space of antisymmetric bilinear functions, find the product representation $\mathbf{x}^1 \wedge \mathbf{x}^2$ of *bivectors* with respect to the vectors $\mathbf{x}^1$, $\mathbf{x}^2$, respectively. Or *given* the linear map $\alpha : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ called "join" which was subject to the *group axioms*, find the *inverse map* $\Delta : \mathbb{X} \to \mathbb{X} \times \mathbb{X}$. The answer to the problem

"*find the normal form of antisymmetric bilinear function*

$$\sum_{1=ij}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} + \frac{1}{2!} \sum_{i,j=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} =$$

$$= \mathbf{x}^1 \wedge \mathbf{x}^2 + \cdots + \mathbf{x}^{r-1} \wedge \mathbf{x}^r \in \mathbb{A}^2 = \mathbf{\Lambda}^2(\mathbb{X}^*)$$

decomposed into the product sum or $r/2$ bivectors *where r is the rank of the antisymmetric bilinear form*"

will be given constructively. The general problem of the decomposition of an *antisymmetric multilinear function* as an element of $\mathbb{A}^p = \mathbf{\Lambda}^p(\mathbb{X}^*)$, $n = \dim \mathbb{X}^*$, into the product sum of *p-vectors* is afterwards obvious.

Case: $\boldsymbol{f}_2 \in \mathbb{A}^2 = \mathbf{\Lambda}^2(\mathbb{X}^*)$, $p = 2$, $n = \dim \mathbb{X} = 2$, $\dim \mathbb{A}^2 = \binom{n}{p} = 1$

The simplest case of an antisymmetric bilinear function $\boldsymbol{f}_2 \in \mathbb{A}^2 = \boldsymbol{\Lambda}^2(\mathbb{X}^*)$ in $n = \dim \mathbb{X} = 2$ dimensions decomposed into a *bivector* is solved by the *change of basis*

$$\mathbf{x}^1 := \mathbf{e}^1 \quad \mathbf{x}^2 := \mathbf{e}^2 f_{12}$$

such that

$$\sum_{1=ij}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} = \mathbf{e}^1 \wedge \mathbf{e}^2 f_{12} = \mathbf{x}^1 \wedge \mathbf{x}^2 . \qquad \clubsuit$$

Case: $\boldsymbol{f}_2 \in \mathbb{A}^2 = \boldsymbol{\Lambda}^2(\mathbb{X}^*),\ p = 2,\ n = \dim \mathbb{X} = 3,\ \dim \mathbb{A}^2 = \binom{n}{p} = 3$

The next case of an antisymmetric bilinear function $\boldsymbol{f}_2 \in \mathbb{A}^2 = \boldsymbol{\Lambda}^2(\mathbb{X}^*)$ in $n = \dim \mathbb{X} = 3$ dimensions decomposed into a *bivector* is solved by the *change of basis*

$$\mathbf{x}^1 := \mathbf{e}^1 - \mathbf{e}^3 f_{23} / f_{12} = \sum_{k_1=1}^{n=3} \mathbf{e}^{k_1} a^1_{k_1}, \mathbf{x}^2 := \mathbf{e}^2 f_{12} - \mathbf{e}^3 f_{13} = \sum_{k_2=1}^{n=3} \mathbf{e}^{k_2} a^2_{k_2}$$

in case of $f_{12} \neq 0$ such

$$\sum_{1=i<j}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} = \mathbf{e}^1 \wedge \mathbf{e}^2 f_{12} + \mathbf{e}^1 \wedge \mathbf{e}^3 f_{13} + \mathbf{e}^2 \wedge \mathbf{e}^3 f_{23} = \mathbf{x}^1 \wedge \mathbf{x}^2 .$$

From the matrix representation of the *change of basis*

$$[\mathbf{x}^1, \mathbf{x}^2] = [\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3] \begin{bmatrix} a^1_1 & a^2_1 \\ a^1_2 & a^2_2 \\ a^1_3 & a^2_3 \end{bmatrix} \textit{or } \mathbf{x} = \mathbf{eA},\ \dim \mathbb{R}(\mathbf{A}) = 2$$

we read the *rank* $r = \dim \mathbb{R}(\mathbf{A}) = 2$ (the dimension of the column space of the matrix **A**) of the antisymmetric bilinear function $\boldsymbol{f}_2(n = 3)$. We recognize $r/2 = 1$, that is *one* factor which leads to the *canonical form* of $\boldsymbol{f}_2(n = 3)$.

Case: $\boldsymbol{f}_2 \in \mathbb{A}^2 = \boldsymbol{\Lambda}^2(\mathbb{X}^*),\ p = 2,\ n = \dim \mathbb{X} = 4,\ \dim \mathbb{A}^2 = \binom{n}{p} = 6$

While the decomposition of $\boldsymbol{f}_2(n = 2)$ and $\boldsymbol{f}_2(n = 3)$ into *one* bivector was trivial, the first interesting case $\boldsymbol{f}_2(n = 4)$ appears now. The reduction scheme of product sums begins with the *first step*: We aim at generating a *first* antisymmetric bilinear function *which excludes* $\{\mathbf{e}^1, \mathbf{e}^2\}$ *from the rest*.

1st step (remove $\mathbf{e}^1$, $\mathbf{e}^2$)

$$\sum_{1=i<j}^{n=4} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} = \mathbf{e}^1 \wedge \mathbf{e}^2 f_{12} + \mathbf{e}^1 \wedge \mathbf{e}^3 f_{13} + \mathbf{e}^1 \wedge \mathbf{e}^4 f_{14}$$

$$+\mathbf{e}^2 \wedge \mathbf{e}^3 f_{23} + \mathbf{e}^2 \wedge \mathbf{e}^4 f_{24} + \mathbf{e}^3 \wedge \mathbf{e}^4 f_{34}$$

$$f_{12} \neq 0$$

$$\mathbf{x}^1 := \mathbf{e}^1 - \mathbf{e}^3 f_{23} / f_{12} - \mathbf{e}^4 f_{24} / f_{12} = \sum_{k_1=1}^{n=4} \mathbf{e}^{k_1} a^1_{k_1}$$

$$\mathbf{x}^2 := \mathbf{e}^2 f_{12} + \mathbf{e}^3 f_{13} = \sum_{k_2=1}^{n=4} \mathbf{e}^{k_2} a^2_{k_2}$$

$$\mathbf{x}^1 \wedge \mathbf{x}^2 = (\mathbf{e}^1 - \mathbf{e}^3 f_{23} / f_{12} - \mathbf{e}^4 f_{24} / f_{12}) \wedge (\mathbf{e}^2 f_{12} + \mathbf{e}^3 f_{13}) =$$

$$= \mathbf{e}^1 \wedge \mathbf{e}^2 f_{12} + \mathbf{e}^1 \wedge \mathbf{e}^3 f_{13} + \mathbf{e}^1 \wedge \mathbf{e}^4 f_{14} + \mathbf{e}^2 \wedge \mathbf{e}^3 f_{23} + \mathbf{e}^2 \wedge \mathbf{e}^4 f_{24} +$$

$$+\mathbf{e}^3 \wedge \mathbf{e}^4 (f_{13} f_{24} - f_{14} f_{23}) / f_{12}$$

$$\sum_{1=i<j}^{n=4} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} = \mathbf{x}^1 \wedge \mathbf{x}^2 + \mathbf{e}^3 \wedge \mathbf{e}^4 [f_{23} - (f_{13} f_{24} - f_{14} f_{23}) / f_{12}].$$

Indeed we have achieved by the *first chance of basis* an antisymmetric *residual* bilinear function which is independent of $\{\mathbf{e}^1, \mathbf{e}^2\}$. The *second step* aims at the same generic scheme.

2nd step (remove $\mathbf{e}^3$, $\mathbf{e}^4$)

$$\mathbf{x}^3 := \mathbf{e}^3 = \sum_{k_3=1}^{n=4} \mathbf{e}^{k_3} a^3_{k_3}, \ \mathbf{x}^4 := \mathbf{e}^4 [f_{34} + (f_{14} f_{23} - f_{13} f_{24}) / f_{12}] = \sum_{k_4=1}^{n=4} \mathbf{e}^{k_4} a^4_{k_4}$$

such that

$$\sum_{1=i<j}^{n=4} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} = \mathbf{x}^1 \wedge \mathbf{x}^2 + \mathbf{x}^3 \wedge \mathbf{x}^4$$

From the matrix representation of the *change of basis*

$$[\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4] = [\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \mathbf{e}^4] \begin{bmatrix} a^1_1 & a^2_1 & a^3_1 & a^4_1 \\ a^1_2 & a^2_2 & a^3_2 & a^4_2 \\ a^1_3 & a^2_3 & a^3_3 & a^4_3 \\ a^1_4 & a^2_4 & a^3_4 & a^4_4 \end{bmatrix}$$

$$\textit{or } \mathbf{x} = \mathbf{eA}, \ \dim \mathbb{R}(\mathbf{A}) = 2$$

subject to

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & f_{12} & 0 & 0 \\ -f_{23}/f_{12} & f_{13} & 1 & 0 \\ 0 & 0 & 0 & f_{34} + (f_{14}f_{23} - f_{13}f_{24})/f_{12} \end{bmatrix} = \mathbf{B}^{-1}$$

$$\sum_{1=i,j}^{n=4} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} = \frac{1}{2!} \sum_{i,j,k,l=1}^{n=4} \mathbf{x}^k \wedge \mathbf{x}^\ell b_k^i b_\ell^j f_{ij} = \sum_{1,k,\ell}^{n=4} \mathbf{x}^k \wedge \mathbf{x}^\ell s_{k\ell}$$

subject to

$$\mathbf{BFB}^* = \mathbf{S}$$

$$\mathbf{S} := [s_{kl}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \text{ (“skew”)}$$

we read the *rank* $r = \dim \mathbb{R}(\mathbf{A}) = 4$ (the dimension of the column space of the matrix **A**) of the antisymmetric bilinear function $\boldsymbol{f}_2(n=4)$. We recognize r / 2 = 2, that is *two* factor which leads to the *canonical form* of $\boldsymbol{f}_2(n=4)$.

Case: $\boldsymbol{f}_2 \in \mathbb{A}^2 = \boldsymbol{\Lambda}^2(\mathbb{X}^*),\ p=2,\ n = \dim \mathbb{X},\ \dim \mathbb{A}^2 = \binom{n}{p}$

Let us begin with the first step of the reduction scheme of product sums by generating a *first* antisymmetric bilinear function *which* excludes $\{\mathbf{e}^1, \mathbf{e}^2\}$ from the rest.

1st step (remove $\mathbf{e}^1$, $\mathbf{e}^2$)

$$\begin{aligned} \sum_{1=i<j}^{n=\dim\mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} = {} & \mathbf{e}^1 \wedge \mathbf{e}^2 f_{12} + \mathbf{e}^1 \wedge \mathbf{e}^3 f_{13} + \cdots + \mathbf{e}^1 \wedge \mathbf{e}^n f_{1n} + \\ & + \mathbf{e}^2 \wedge \mathbf{e}^3 f_{23} + \mathbf{e}^2 \wedge \mathbf{e}^4 f_{24} + \cdots + \mathbf{e}^2 \wedge \mathbf{e}^n f_{2n} + \\ & + \mathbf{e}^3 \wedge \mathbf{e}^4 f_{34} + \mathbf{e}^3 \wedge \mathbf{e}^5 f_{35} + \cdots + \mathbf{e}^3 \wedge \mathbf{e}^n f_{3n} + \\ & + \cdots + \mathbf{e}^{n-1} \wedge \mathbf{e}^n f_{n-1n} \\ & f_{12} \neq 0 \end{aligned}$$

$$\mathbf{x}^1 := \mathbf{e}^1 - \mathbf{e}^3 f_{23}/f_{12} - \cdots - \mathbf{e}^n f_{2n}/f_{12} = \sum_{k_1=1}^{n} \mathbf{e}^{k_1} a_{k_1}^1 \,,$$

$$\mathbf{x}^2 := \mathbf{e}^2 f_{12} + \mathbf{e}^3 f_{13} + \cdots + \mathbf{e}^n f_{1n} = \sum_{k_2=1}^{n} \mathbf{e}^{k_2} a_{k_2}^2 \,,$$

$$\mathbf{x}^1 \wedge \mathbf{x}^2 = \mathbf{e}^1 \wedge \mathbf{e}^2 f_{12} + \mathbf{e}^1 \wedge \mathbf{e}^3 f_{13} + \cdots + \mathbf{e}^1 \wedge \mathbf{e}^n f_{1n} -$$
$$-\mathbf{e}^3 \wedge \mathbf{e}^2 f_{23} - \mathbf{e}^3 \wedge \mathbf{e}^4 f_{23} f_{14} / f_{12} - \cdots - \mathbf{e}^3 \wedge \mathbf{e}^n f_{23} f_{1n} / f_{12}$$
$$-\mathbf{e}^n \wedge \mathbf{e}^2 f_{24} - \mathbf{e}^n \wedge \mathbf{e}^3 f_{2n} f_{13} / f_{12} - \cdots -$$
$$-\mathbf{e}^{n-1} \wedge \mathbf{e}^n f_{2n-1} f_{1n} / f_{12}$$

$$\sum_{1=i<j}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} = \mathbf{x}^1 \wedge \mathbf{x}^2 + \mathbf{e}^3 \wedge \mathbf{e}^4 [f_{34} + (f_{23} f_{14} - f_{24} f_{13}) / f_{12}] +$$
$$+ \cdots + \mathbf{e}^3 \wedge \mathbf{e}^n [f_{3n} + (f_{23} f_{1n} - f_{2n} f_{13}) / f_{12}] + \cdots +$$
$$+ \mathbf{e}^{n-1} \wedge \mathbf{e}^n [f_{n-1n} + (f_{2n-1} f_{1n} - f_{2n} f_{1n-1}) / f_{12}]$$

$$\sum_{1=i<j}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} = \mathbf{x}^1 \wedge \mathbf{x}^2 + \mathbf{e}^3 \wedge \mathbf{e}^4 f'_{34} + \cdots +$$
$$+ \mathbf{e}^3 \wedge \mathbf{e}^n f'_{3n} + \cdots + \mathbf{e}^{n-1} \wedge \mathbf{e}^n f'_{n-1n}$$

subject to

$$f'_{34} := f_{34} + (f_{23} f_{14} - f_{24} f_{13}) / f_{12},$$
$$f'_{3n} := f_{3n} + (f_{23} f_{1n} - f_{2n} f_{13}) / f_{12}, \cdots,$$
$$f'_{n-1n} := f_{n-1n} + (f_{2n-1} f_{1n} - f_{2n} f_{1n-1}) / f_{12}.$$

The *second step* of the function scheme of product sums generates a *second* antisymmetric bilinear function which exclude $\{\mathbf{e}^3, \mathbf{e}^4\}$ from the rest.

2nd step (remove $\mathbf{e}^3$, $\mathbf{e}^4$)

$$f'_{34} \neq 0$$

$$\mathbf{x}^3 := \mathbf{e}^3 - \mathbf{e}^5 f'_{45} / f'_{34} - \cdots - \mathbf{e}^n f'_{4n} / f'_{34} = \sum_{k_3=1}^{n} \mathbf{e}^{k_3} a^3_{k_3}$$

$$\mathbf{x}^4 := \mathbf{e}^4 f'_{23} + \mathbf{e}^5 f'_{35} + \cdots + \mathbf{e}^n f'_{3n} = \sum_{k_4=1}^{n} \mathbf{e}^{k_4} a^4_{k_4}$$

$$\mathbf{x}^3 \wedge \mathbf{x}^4 = \mathbf{e}^3 \wedge \mathbf{e}^4 f'_{34} + \mathbf{e}^3 \wedge \mathbf{e}^5 f'_{35} + \cdots + \mathbf{e}^3 \wedge \mathbf{e}^n f_{3n} -$$
$$-\mathbf{e}^5 \wedge \mathbf{e}^4 f'_{45} - \mathbf{e}^5 \wedge \mathbf{e}^6 f'_{45} f'_{36} / f'_{34} - \cdots - \mathbf{e}^5 \wedge \mathbf{e}^n f'_{45} f'_{3n} / f'_{34} - \cdots -$$
$$-\mathbf{e}^{n-1} \wedge \mathbf{e}^n f'_{4n-1} f'_{3n} / f'_{34}.$$

$$\sum_{1=i<j}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} = \mathbf{x}^1 \wedge \mathbf{x}^2 + \mathbf{x}^3 \wedge \mathbf{x}^4 +$$
$$+ \mathbf{e}^5 \wedge \mathbf{e}^6 [f'_{56} + (f'_{45} f'_{36} - f'_{46} f'_{35}) / f'_{34}] + \cdots +$$
$$+ \mathbf{e}^5 \wedge \mathbf{e}^n [f'_{5n} + (f'_{45} f'_{3n} - f'_{4n} f'_{35}) / f'_{34}] + \cdots +$$
$$+ \mathbf{e}^{n-1} \wedge \mathbf{e}^n [f'_{n-1n} + (f'_{4n-1} f'_{3n} - f'_{4n} f'_{3n-1}) / f'_{34}].$$

For a given $n = \dim \mathbb{X}^*$ the reduction machinery stops when we have a-chieved the final aim of the complete reduction

$$\sum_{1=i<j}^{n=\dim \mathbb{X}^*} \mathbf{e}^i \wedge \mathbf{e}^j f_{ij} =$$
$$= \mathbf{x}^1 \wedge \mathbf{x}^2 + \mathbf{x}^3 \wedge \mathbf{x}^4 + \cdots + \mathbf{x}^{r-3} \wedge \mathbf{x}^{r-2} + \mathbf{x}^{r-1} \wedge \mathbf{x}^r \in \mathbb{A}^2 = \mathbf{\Lambda}(\mathbb{X}^*).$$

The rank $r = \dim \mathbb{R}(\mathbf{A})$ will decide upon the number r / 2 of factors in the product sum decomposition of the antisymmetric bilinear function. In its normal form the antisymmetric matrix $[f_{ij}] =: \mathbf{F} \in \mathbb{R}^{n\times n}$ has been transformed into the *block antisymmetric* (“*skew*”) *matrix* $[s_{kl}] =: \mathbf{S} \in \mathbb{R}^{r\times r}$:

$$\mathbf{S} := [s_{kl}] = \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \\ & & \boxed{\cdots} & \\ & & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} \end{bmatrix} \in \mathbb{R}^{r\times r}$$

The reference to *symplectic geometry* is obvious.

The genesis of the normal form of the antisymmetric multilinear function

$$\sum_{1<i_1<\cdots<i_p}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_p} f_{i_1\cdots i_p} =$$
$$= \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^p + \cdots + \mathbf{x}^{r-(p+1)} \wedge \cdots \wedge \mathbf{x}^r \in \mathbb{A}^p = \mathbf{\Lambda}^p(\mathbb{X}^*)$$

namely its decomposition into the *product sum* of *p-vectors* follows similar patterns as being outlined for the antisymmetric bilinear function $\boldsymbol{f}_2 \in \mathbb{A}^2 = \mathbf{\Lambda}^2(\mathbb{X}^*)$, $\dim \mathbb{X}^* = n$.

♣

Historical Aside

For a historical perspective for the generation of a product sum decomposition of an antisymmetric bilinear function into bivectors we refer to *J. Zund*: The theory of bivectors, Tensor New Series *22* (1971) 179-185. Note that $\mathbf{x}^1 \wedge \mathbf{x}^2, \mathbf{x}^3 \wedge \mathbf{x}^4 \cdots, \mathbf{x}^{r-1} \wedge \mathbf{x}^r$ are called *blades*. For other details we refer to *A. Crumeyrolle* (1990 p. 30-31) and *M. Marcus* (1975, Part II, p. 1-10).

♣

3-3 Interior Algebra

Now we continue with *interior algebra*.

Definition 3-2 (symmetric algebra, interior algebra, algebra of symmetric multilinear functions):

In terms of general coordinate base $\{\mathbf{b}^1, \cdots, \mathbf{b}^n\} = \mathrm{span}\,\mathbb{X}^*$ *let* $f \in \mathbb{R}^p$, $g \in \mathbb{R}^r$, $h \in \mathbb{R}^t$ *be* symmetric *multilinear functions on a linear space* $\mathbb{X}$*, namely*

$$f = \frac{1}{p!} \sum_{i_1, \cdots, i_p = 1}^{n = \dim \mathbb{X}} \mathbf{b}^{i_1} \vee, \cdots, \vee\, \mathbf{b}^{i_p} f_{i_1 \cdots i_p} \in \mathbb{R}^p(\mathbb{X}^*) ,$$

$$g = \frac{1}{r!} \sum_{j_1, \cdots, j_r = 1}^{n = \dim \mathbb{X}} \mathbf{b}^{j_1} \vee, \cdots, \vee\, \mathbf{b}^{j_r} g_{j_1 \cdots j_r} \in \mathbb{R}^r(\mathbb{X}^*) ,$$

$$h = \frac{1}{t!} \sum_{k_1, \cdots, k_t = 1}^{n = \dim \mathbb{X}} \mathbf{b}^{k_1} \vee, \cdots, \vee\, \mathbf{b}^{k_t} h_{k_1 \cdots k_t} \in \mathbb{R}^t(\mathbb{X}^*) .$$

A <u>*symmetric multilinear algebra*</u> *over* $\mathbb{R}$ *(also called interior algebra) as a* $\mathbb{R}$*-algebra consists of a* ^{p}, <u>*two international relations*</u> *(addition and inner multiplication)* $+ : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$, $\vee : \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}^{p \times r}$, *and* <u>*one external relation*</u> *(external multiplication)* $\times : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^p$ *where the following properties hold*

first: addition

f, g, h $\in \mathbb{R}^p$, "+" (internal relation of type addition)

(**G**1+) $(f + g) + h = f + (g + h)$ (*associativity of type addition*)

(**G**2+) $f + 0 = f$ (*identity of addition*)

(**G**3+) $f - f = 0$ (*inverse of addition*)

(**G**4+) $f + g = g + f$ (*commutativity of addition*)

second: multiplication

f, g $\in \mathbb{R}^p$, r, s $\in \mathbb{R}$ "×" (external relation of type multiplication)

(**D**1+) $r \times (f + g) = r \times f + r \times g$ (1*st distributivity*)

(**D**2+) $(r + s) \times f = r \times f + s \times f$ (2*nd distributivity*)

(**D**3+) $1 \times f = f$

third: exterior product

$f \in \mathbb{R}^p$, $g \in \mathbb{R}^r$, $h \in \mathbb{R}^t$, "$\vee$" (*internal relation of type exterior product*)

$$f \vee g = \frac{1}{p!r!} \sum_{i_1,\cdots,i_p,i_{p+1},\cdots,i_{p+r}}^{n=\dim X^*} \mathbf{b}^{i_1} \vee \cdots \vee \mathbf{b}^{i_p} \vee \mathbf{b}^{i_{p+1}} \vee \cdots \vee \mathbf{b}^{i_{p+r}} f_{i_1\cdots i_p} g_{i_{p+1}\cdots i_{p+r}}$$

(**G**1$\vee$) $(f \vee g) \vee h = f \vee (g \vee h)$
(*associativity of internal multiplication of type interior product*)

(**D**1$\vee +$) $f \vee (g+h) = f \vee g + f \vee h$, *if* $f \in \mathbf{S}^p$, $g, h \in \mathbf{S}^r$
(*additive distributivity w. r. t. internal multiplications of type interior product*)

(**D**2$\vee +$) $(f+g) \vee h = f \vee h + g \vee h$, *if* $f, g \in \mathbf{S}^p, h \in \mathbf{S}^r$
(*additive distributivity w. r. t. internal multiplications of type interior product*)

(**D**3$\vee \times$) $r \times (f \vee g) = (r \times f) \vee g$

(*distributivity of internal multiplication of type interior product and external multiplication*)

(**G**4$\vee$) $g \vee h = h \vee g$.
(*commutativity of internal multiplication of type interior product*)

Corollary 3-3:

$$\dim \mathbb{S}^p = \binom{n+p-1}{p}$$

$$\dim \oplus_{p=0}^{n} \mathbb{S}^p = \dim \oplus_{p=0}^{n} \vee \mathbb{S}^p = \sum_{p=0}^{n=\dim \mathbb{X}^*} \frac{1}{p!} n(n+1)\cdots(n+p-1) .$$

Scholia

References to *multilinear algebra*, in particular to *the Hodge star dualizer*, are *P. Bamberg and S. Sterberg* (19), *M. Barnabei et al* (1985), *G. Berman* (1961), *A. Crumeyrolle* (1990), *W. H. Greub* (1967), *E. Lamberch* (1993) and *M. Marcus* (1975).

Chapter 4

Clifford algebra

We already took advantage of the notion of *Clifford algebra*. Here we finally confront you with the definition of "*orthogonal Clifford algebra* $C\ell(p,\, q)$". But on our way to *Clifford algebra* we have to generalize at first the notion of a basis, in particular its bilinear form.

Theorem 4-1 (bilinear form):

Suppose that the <u>*bracket*</u> $<\cdot|\cdot>$ *or* $g(\cdot,\cdot): \mathbb{X}\times\mathbb{X}^* \to \mathbb{R}$ *is a bilinear form a finite dimensional linear space* $\mathbb{X}$*, e.g. a vector space, over the field* $\mathbb{R}$ *of real numbers, in addition* $\mathbb{X}^*$ *its dual space such that n* = dim $\mathbb{X}^*$= dim $\mathbb{X}$. *There exists a basis* $\{\mathbf{e}_1,\cdots,\mathbf{e}_n\}$ *such that*

(i) $$<\mathbf{e}_i\,|\,\mathbf{e}_i>=0 \text{ or } g(\mathbf{e}_i,\mathbf{e}_j)=0 \text{ for } i\neq j$$

(ii) $$\begin{cases} <\mathbf{e}_{i_1}\,|\,\mathbf{e}_{i_1}>=+1 \; or \; g(\mathbf{e}_{i_1},\mathbf{e}_{i_1})=+1 \;\; for \; 1\leq i_1<p, \\ <\mathbf{e}_{i_2}\,|\,\mathbf{e}_{i_2}>=-1 \; or \; g(\mathbf{e}_{i_2},\mathbf{e}_{i_2})=-1 \;\; for \; p+1\leq i_2<p+q=r, \\ <\mathbf{e}_{i_3}\,|\,\mathbf{e}_{i_3}>=0 \;\; or \; g(\mathbf{e}_{i_3},\mathbf{e}_{i_3})=0 \;\; for \; r+1\leq i_3<n \end{cases}$$

holds.

The numbers *r* and *p* are determined exclusively by the bilinear form. *r* is called the *rank*, *r* – *p* = *q* is called the *index* and the *ordered pair* (*p*, *q*) the *signature*. The theorem assures that any two spaces of the same dimension with *bilinear forms of the same signature* are isometrically isomorphic. A *scalar product* ("inner product") in this context is a *non degenerate bilinear form, i.e., a form with rank equal to the dimension of* $\mathbb{X}$. When dealing with low dimensional spaces as we do, we will often indicate *the signature* with a series of plus and minus signs and zeroes where appropriate. For example, the signature of $\mathbb{R}^4_1$ my be written (+++-) instead of (3, 1). If the bilinear form is *non degenerate*, a basis with the properties listed in *Theorem A16* is called an *orthonormal basis* ("unimodular") for $\mathbb{X}$ with respect to the bilinear form.

Definition 4-1 (orthogonal Clifford algebra $C\ell(p,\ q)$):

The <u>*orthogonal Clifford algebra*</u> $C\ell(p,\ q)$ *is the algebra of polynomials generated by the* <u>*direct sum*</u> *of the space of multilinear functions*

$$\oplus_{m=0}^{n} \overset{*}{\wedge}{}^{m} (\mathbb{X}^{*})$$

on a linear space $\mathbb{X}$*, respectively its dual* $\mathbb{X}^{*}$ *over the field of real numbers* $\mathbb{R}_p^n$ *of dimension*

$$\dim \oplus_{m=0}^{n} \overset{*}{\wedge}{}^{m} (\mathbb{X}^{*}) = 2^{n}$$

and signature (p, q)*, namely*

$$\mathbf{1} f_0 + \sum_{i_1=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} f_{i_1} + \sum_{i_1,i_2=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \overset{*}{\wedge} \mathbf{e}^{i_2} f_{i_1 i_2} + \\ + \sum_{i_1,i_2,i_3=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \overset{*}{\wedge} \mathbf{e}^{i_2} \overset{*}{\wedge} \mathbf{e}^{i_3} f_{i_1 i_2 i_3} + \cdots + \sum_{i_1,\cdots,i_n=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \overset{*}{\wedge} \cdots \overset{*}{\wedge} \mathbf{e}^{i_n} f_{i_1 \cdots i_n}$$

subject to the <u>*Clifford product*</u>*, also called the* <u>*Clifford dualizer*</u>*,*

(i) $\mathbf{e}_i \overset{*}{\wedge} \mathbf{e}_j = -\mathbf{e}_j \overset{*}{\wedge} \mathbf{e}_i \quad$ *for* $i \neq j$

(ii)
$$\begin{bmatrix} \mathbf{e}_{i_1} \overset{*}{\wedge} \mathbf{e}_{i_1} = g(\mathbf{e}_{i_1}, \mathbf{e}_{i_1}) = +1 \ \text{for } 1 \le i_1 < p \\ \mathbf{e}_{i_2} \overset{*}{\wedge} \mathbf{e}_{i_2} = g(\mathbf{e}_{i_2}, \mathbf{e}_{i_2}) = -1 \ \text{for } p+1 \le i_2 < p+q = r \\ \mathbf{e}_{i_3} \overset{*}{\wedge} \mathbf{e}_{i_3} = g(\mathbf{e}_{i_3}, \mathbf{e}_{i_3}) = \quad 0 \ \text{for } r+1 \le i_3 < n, \end{bmatrix}$$

or

$$\mathbf{e}_1 \overset{*}{\wedge} \mathbf{e}_j + \mathbf{e}_j \overset{*}{\wedge} \mathbf{e}_i = 2g(\mathbf{e}_i, \mathbf{e}_j)\delta_{ij}$$

subject to

$$\begin{bmatrix} g(\mathbf{e}_{i_1}, \mathbf{e}_{i_1}) = +1 \ \text{for } 1 \le i_1 < p \\ g(\mathbf{e}_{i_2}, \mathbf{e}_{i_2}) = -1 \ \text{for } p+1 \le i_2 < p+q = r \\ g(\mathbf{e}_{i_3}, \mathbf{e}_{i_3}) = \quad 0 \ \text{for } r+1 \le i_3 < n, \end{bmatrix}$$

1 *being the* <u>*neutral element*</u>*. If* $\mathbf{e}_k \overset{*}{\wedge} \mathbf{e}_k = 0$ *or* $g(\mathbf{e}_k, \mathbf{e}_k) = 0$ *holds uniformly the* <u>*orthogonal Clifford algebra*</u> $C\ell(p,\ q)$ *reduces to the polynomial algebra of* <u>*antisymmetric multilinear functions*</u>

$$\oplus_{m=0}^{n} \mathbf{A}^{m} = \oplus_{m=0}^{n} \mathbf{\Lambda}^{m} (\mathbb{X}^{*}) = \mathbf{\Lambda}(\mathbb{X}^{*})$$
$$\dim \oplus_{m=0}^{n} \mathbf{A}^{m} = \dim \oplus_{m=0}^{n} \mathbf{\Lambda}^{m} (\mathbb{X}^{*}) = \dim(\mathbb{X}^{*}) = 2^{n}.$$

represented by

$$\mathbf{1} f_0 + \frac{1}{1!} \sum_{i_1=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} f_{i_1} + \frac{1}{2!} \sum_{i_1,i_2=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} f_{i_1 i_2} + \\ + \frac{1}{3!} \sum_{i_1,i_2,i_3=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \mathbf{e}^{i_3} f_{i_1 i_2 i_3} + \cdots + \frac{1}{n!} \sum_{i_1,\cdots,i_n=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_n} f_{i_1 \cdots i_n}.$$

***Example 4-1*: Clifford product: $\mathbf{x} * \mathbf{y}$, $\mathbf{x} * \mathbf{y} \in \mathbb{X}$, *sign* $\mathbb{X} = (3,0)$**

Let $\mathbf{x} * \mathbf{y} \in \mathbb{X}$ be a real three-dimensional vector space $\mathbb{X}$ of signature (3, 0). Then $\mathbf{x} * \mathbf{y}$ (read: "**x***Clifford***y**") with respect to a set of the bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ accounts for

$$\mathbf{x} \overset{*}{\wedge} \mathbf{y} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{e}_i \overset{*}{\wedge} \mathbf{e}_j x^i y^j = \sum_{i=1}^{3}\sum_{j=1}^{3} x^i y^j \mathbf{e}_i \overset{*}{\wedge} \mathbf{e}_j$$

$$\mathbf{x} \overset{*}{\wedge} \mathbf{y} = (\mathbf{e}_1 x^1 + \mathbf{e}_2 x^2 + \mathbf{e}_3 x^3) \overset{*}{\wedge} (\mathbf{e}_1 y^1 + \mathbf{e}_2 y^2 + \mathbf{e}_3 y^3)$$

$$\mathbf{x} \overset{*}{\wedge} \mathbf{y} = \mathbf{1}(x^1y^1 + x^2y^2 + x^3y^3) + \mathbf{e}_1 \wedge \mathbf{e}_2 (x^1y^2 - x^2y^1) + \\ + \mathbf{e}_2 \wedge \mathbf{e}_3 (x^2y^3 - x^3y^2) + \mathbf{e}_3 \wedge \mathbf{e}_1 (x^3y^1 - x^1y^3)$$

Indeed $\mathbf{x} \overset{*}{\wedge} \mathbf{y}$ as a *Clifford number* is decomposed into a scalar part and an antisymmetric tensor part with respect to the bilinear basis $\{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1\}$.

Tensor algebra or the algebra of multilinear functions, namely

$$\mathbf{1} f_0 + \sum_{i_1=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} f_{i_1} + \sum_{i_1,i_2=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \otimes \mathbf{e}^{i_2} f_{i_1 i_2} + \\ + \sum_{i_1,i_2,i_3=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \otimes \mathbf{e}^{i_2} \otimes \mathbf{e}^{i_3} f_{i_1 i_2 i_3} + \cdots + \sum_{i_1,\cdots,i_n=1}^{n=\dim \mathbb{X}^*} \mathbf{e}^{i_1} \otimes \cdots \otimes \mathbf{e}^{i_n} f_{i_1 \cdots i_n} \in \\ \in \oplus_{m=0}^{n} \otimes (\mathbb{X}^*) = \otimes(\mathbb{X}) = \boldsymbol{T}^+ \oplus \boldsymbol{T}^-$$

$$\textit{subject to} \begin{bmatrix} \boldsymbol{T}^+ = \oplus_{h=0} \otimes^{2h} (\mathbb{X}) \ \ (\text{"}\textit{even}\text{"}) \\ \boldsymbol{T}^- = \oplus_{k=0} \otimes^{2k+1} (\mathbb{X}) \ \ (\text{"}\textit{odd}\text{"}) \end{bmatrix}$$

in the sum of two spaces, $\boldsymbol{T}^+$ and $\boldsymbol{T}^-$, respectively, in particular

$$C\ell^+ \ni \mathbf{1} f_0 + \frac{1}{2!}\sum_{i_1,i_2=1}^{n} \mathbf{e}^{i_1} \overset{*}{\wedge} \mathbf{e}^{i_2} f_{i_1 i_2} + \frac{1}{4!}\sum_{i_1,i_2,i_3,i_4=1}^{n} \mathbf{e}^{i_1} \overset{*}{\wedge} \mathbf{e}^{i_2} \overset{*}{\wedge} \mathbf{e}^{i_3} \overset{*}{\wedge} \mathbf{e}^{i_4} f_{i_1 i_2 i_3 i_4} + \cdots,$$

$$C\ell^- \ni \frac{1}{1!}\sum_{i_1=1}^{n} \mathbf{e}^{i_1} f_{i_1} + \frac{1}{3!}\sum_{i_1,i_2,i_3=1}^{n} \mathbf{e}^{i_1} \overset{*}{\wedge} \mathbf{e}^{i_2} \overset{*}{\wedge} \mathbf{e}^{i_3} f_{i_1 i_2 i_3} + \cdots.$$

Obviously $C\ell^+$ as well as $C\ell^-$ are *subalgebras* of $C\ell$. Let the *Clifford numbers* z be divided into $z^+ \in C\ell^+$ and $z^- \in C\ell^-$, then the properties

$$z^+ \overset{*}{\wedge} z^+ \in C\ell^+, \ z^- \overset{*}{\wedge} z^- \in C\ell^+, \ z^+ \overset{*}{\wedge} z^- \in C\ell^-$$

prove that $C\ell(p,\ q)$ is graded over the *cydric group* $\mathbb{Z}_2 = \{0,1\}$.

Chapter 5
Partial contraction of tensor-valued function

While the Hodge star operator constituted a linear map of *antisymmetric multilinear functions* $f \in \mathbf{A}^p \subset \mathbf{T}^p$ into *antisymmetric multilinear functions* $* f \in \mathbf{A}^{n+p} \subset \mathbf{T}^{n-p}$ there is a similar linear map called *partial contraction* which transforms multilinear functions $f \in \mathbf{T}^p$ into multilinear functions $\mathrm{c}\ \ f \in \mathbf{T}^{p-s}$ (read contraction *f*),

$$\mathrm{c}_{rs}\,\mathbf{T}^p \ni f = \{\sum_{i_1,\ldots,i_p}^{\dim \mathbb{X}^*} \mathbf{e}^{i_1} \otimes \cdots \otimes \mathbf{e}^{i_p} f_{i_1 \cdots i_p}\} \Rightarrow$$

$$\{\sum_{k=1}^{n} \sum_{i_1,\cdots,i_{r-1},i_{r+1},\cdots,i_{s-1},i_{s+1},\cdots,i_p=1}^{n}$$

$$\mathbf{e}^{i_1} \otimes \cdots \otimes \mathbf{e}^{i_{r-1}} \otimes \mathbf{e}^{i_{r+1}} \otimes \cdots \otimes \mathbf{e}^{i_{s-1}} \otimes \mathbf{e}^{i_{s+1}} \otimes \cdots \otimes \mathbf{e}^{i_r} f_{i_1 \cdots i_{r+1} \cdots i_{s-1} k i_{s+1} \cdots i_p = 1}\} \in \mathbf{T}^{p-2}$$

or

$$\mathrm{c}_{rs}\ f : f_{i_1 \cdots i_p} \Rightarrow \sum_{k=1}^{n} f_{i_1 \cdots i_{r-1} k i_{r+1} \cdots i_{s-1} k i_{s+1} \cdots i_p}$$

in array notation. Obviously by *partial contraction* the tensor element $f_{\cdots i_r \cdots i_s}$ has been removed by summation. More generally, for a (*p*, *q*) tensor-valued multilinear function $f \in \mathbf{T}_q^p$, $\mathrm{c}_r^s\ f \in \mathbf{T}_{q-1}^{p-1}$ maps linearly into a (p-1, q-1) tensor-valued multilinear function by means of

$$\mathrm{c}_r^s\ f : \mathbf{T}_q^p \ni f = \{\sum_{i_1,\cdots,i_p}^{n} \sum_{j_1,\cdots,j_p}^{n} \mathbf{e}^{i_1} \otimes \cdots \otimes \mathbf{e}^{i_p} \otimes \mathbf{e}_{j_1} \otimes \cdots \otimes \mathbf{e}_{j_p} f_{i_1 \cdots i_p}^{j_1 \cdots j_p}\} \Rightarrow$$

$$\{\sum_{k=1}^{n} \sum_{i_1,\cdots,i_{r-1},i_{r+1},\cdots,i_p}^{n=\dim \mathbb{X}^*} \sum_{j_1,\cdots,j_{s-1},j_{s+1},\cdots,j_p}^{n=\dim \mathbb{X}} \mathbf{e}^{i_1} \otimes \cdots \otimes \mathbf{e}^{i_{r-1}} \otimes \mathbf{e}^{i_{r+1}} \otimes \cdots \otimes \mathbf{e}^{i_p}$$

$$\otimes \mathbf{e}_{j_1} \otimes \cdots \otimes \mathbf{e}_{j_{s-1}} \otimes \mathbf{e}_{j_{s+1}} \otimes \cdots \otimes \mathbf{e}_{j_q} f_{i_1 \cdots i_{s-1},k,i_{s+1} \cdots i_p}^{j_1 \cdots j_{s-1},k,j_{s+1} \cdots j_p}\} \in \mathbf{T}_{q-1}^{p-1}.$$

While a *partial contraction map* reduces both covariant and contravariant degree by one, *successive contraction* define a map down to $\mathbf{T}_0^0 = \mathbb{R}$, but not uniquely. For instance, for an equal covariant and contravariant degree, *n* = *q* the contraction map $\neg\ f$ (*read* key) is defined by

$$\mathbf{T}_p^p \ni f = \{ \sum_{i_1,\cdots,i_p} \sum_{j_1,\cdots,j_p} \mathbf{e}^{i_1} \otimes \cdots \otimes \mathbf{e}^{i_p} \otimes \mathbf{e}_{j_1} \otimes \cdots \otimes \mathbf{e}_{j_p} f_{i_1,\cdots,i_p}^{j_1,\cdots,j_p} \} \Rightarrow$$

$$\Rightarrow \sum_{k_1,\cdots,k_p}^{n} f_{k_1,\cdots,k_p}^{k_1,\cdots,k_p} =: \neg f \in \mathbf{T}_0^0 \in \mathbb{R} .$$

There are p! possible *total contraction* $\mathbf{T}_p^p \to \mathbb{R}$ according to how we pair the elements of $f_{i_1\cdots i_p}^{j_1\cdots j_p}$. Even worse, for different covariant and contravariant degree, $p \geq q \neg f$ generates an element of $\mathbf{T}_{p-q}^0$, namely

$$\mathbf{T}_q^p \ni f = \{ \sum_{i_1,\cdots,i_q,i_{q+1},\cdots,i_p} \sum_{j_1,\cdots,j_p}$$

$$\mathbf{e}^{i_1} \otimes \cdots \otimes \mathbf{e}^{i_q} \otimes \mathbf{e}^{i_{q+1}} \otimes \cdots \otimes \cdots \otimes \mathbf{e}^{i_p} \otimes \mathbf{e}_{j_1} \otimes \cdots \otimes \mathbf{e}_{j_q} f_{i_1,\cdots,i_q,i_{q+1},\cdots,i_p}^{j_1,\cdots,j_q} \} \Rightarrow$$

$$\Rightarrow \neg f =: \sum_{k_1,\cdots,k_q} \sum_{i_{q+1},\cdots,i_p} \mathbf{e}^{i_{q+1}} \otimes \cdots \otimes \mathbf{e}^{i_p} f_{k_1,\cdots,k_q,i_{q+1},\cdots,i_p}^{k_1,\cdots,k_p} \in \mathbf{T}_{p-q}^0$$

or

$$\neg f : f_{i_1,\cdots,i_q,i_{q+1},\cdots,i_p}^{j_1,\cdots,j_p} \to \sum_{k_1,\cdots,k_q} f_{k_1,\cdots,k_q,i_{q+1},\cdots,i_p}^{k_1,\cdots,k_q}$$

namely an array of dimension $\dim \neg f = n \times \cdots \times n(p-q)$ times. Let us continue the decomposition of tensor-valued multilinear functions $f \in \mathbf{T}_q^p = \mathbf{S}_q^p \oplus \mathbf{A}_q^p$ - sometimes written $\wedge_q^p \oplus \vee_q^p$ in order to emphasize the spaces spanned by the interior product "$\vee$"- namely the decomposition of $\mathbf{S}_0^2, \mathbf{S}_1^1, \mathbf{S}_2^0$, respectively into $\mathbf{S}_0^2 = \mathbf{C}_0^2 \oplus \mathbf{D}_0^2$, $\mathbf{S}_1^1 = \mathbf{C}_1^1 \oplus \mathbf{D}_1^1$, $\mathbf{S}_2^0 = \mathbf{C}_2^0 \oplus \mathbf{D}_2^0$, respectively of *contracted symmetric bilinear functions* and *their deviatoric residuals*, also called trace-free. The origin of such an additional decomposition is the following situation: Assume a (2, 0) tensor-valued bilinear function *f* which is decomposed as an element of $\mathbf{T}_0^2 = \mathbf{S}_0^2 \oplus \mathbf{A}_0^2$ (the *direct sum* of $\mathbf{S}_0^2$ and $\mathbf{A}_0^2$), in short $\mathbf{T}^2 = \mathbf{S}^2 \oplus \mathbf{A}^2$. Note the (2, 0) tensor-valued *antisymmetric bilinear function* as an element of $\mathbf{A}^2$ has $\{\mathrm{tr}\, f = 0 \mid f \in \mathbf{A}^2\}$. Accordingly for a (0, 0) tensor-valued bilinear function it is worthwhile to compute $\{\mathrm{tr}\, f \mid f \in \mathbf{S}^2\}$, namely the trace of a (2, 0) tensor-valued *symmetric bilinear function*. Whether or not $\{\mathrm{tr}\, f \mid f \in \mathbf{S}^2\}$ is zero as will be seen later is an important property of a symmetric tensor of type (2, 0). As a constituent of a symmetric (2, 0) tensor $[f_{ij}] = [f_{ji}]$ or $\mathbf{F} = \mathbf{F}^T$

$$\frac{1}{n!}(\mathrm{tr}\, f)[\delta_{ij}] \;or\; \frac{1}{n}(\mathrm{tr}\,\mathbf{F})\mathbf{I}_n \;\; versus \;\; \frac{1}{n}(\mathrm{tr}\, f)[g_{ij}] \;\; or \;\; \frac{1}{n}(\mathrm{tr}\,\mathbf{F})\mathbf{I}_n$$

with respect to an orthonormal base, $\mathrm{span}\,\mathbb{X}^* = \{\mathbf{e}^1, \cdots, \mathbf{e}^n\}$, *versus a set of linear independent bases*, $\mathrm{span}\,\mathbb{X}^* = \{\mathbf{b}^1, \cdots, \mathbf{b}^n\}$, amounts to the *factorization* of $\mathrm{tr}\,\mathbf{F} \in \mathbb{R}$, $\mathrm{tr}\,\mathbf{F}\mathbf{G}^{-1} \in \mathbb{R}$, respectively, also called *scalars* and the matrices of the metric $[\delta_{ij}]$ or $\mathbf{I}_n \in \mathbb{R}^n \times \mathbb{R}^n$, $[g_{ij}] = \mathbf{G} \in \mathbb{R}^n \times \mathbb{R}^n$, respectively. A symmetric (2, 0) tensor $[f_{ij}] = [f_{ji}]$ enjoys the "*contracted decomposition*"

$$[f_{ij}]=[f_{ji}]=\frac{1}{n}\text{tr } f[\delta_{ij}]+[f_{ij}-\frac{1}{n}(\text{tr } f)\delta_{ij}]$$

with respect to an orthonormal base
versus

$$\mathbf{F}=\mathbf{F}^T=\frac{1}{n}(\text{tr}\,\mathbf{F}\mathbf{G}^{-1})\mathbf{G}-[\mathbf{F}-\frac{1}{n}(\text{tr}\,\mathbf{F}\mathbf{G}^{-1})\mathbf{G}],$$

$$f_{ij}=f_{ji}=\frac{1}{2}(\sum_{k,\ell=1}^{n} f_{k\ell}g^{k\ell})g_{ij}-(f_{ij}-\frac{1}{n}(\sum_{k,\ell=1}^{n} f_{k\ell}g^{k\ell})g_{ij})$$

or $\mathbf{S}^2=\mathbf{C}^2\oplus\mathbf{D}^2$ (the direct sum of $\mathbf{C}^2$ and $\mathbf{D}^2$) with $\mathbf{C}^2\ni\text{tr } f[\delta_{ij}]/n$, $\mathbf{C}^2\ni(\text{tr}$ $\mathbf{F}\mathbf{G}^{-1})\mathbf{G}$, $\mathbf{D}^2\ni[f_{ij}-(\text{tr } f)\delta_{ij}/n]$, $\mathbf{D}^2\ni\mathbf{F}-(\text{tr}\,\mathbf{F}\mathbf{G}^{-1})\mathbf{G}/n$ and $\mathbf{S}^2\ni[g_{ij}]$. Due to $\text{tr}[\delta_{ij}]=n$, $\text{tr}[f_{ij}-(\text{tr } f)\delta_{ij}n]=0$, the (2, 0) symmetric tensor

$$[d_{ij}]:=[f_{ij}-\frac{1}{n}(\text{tr } f)\delta_{ij}]$$

with respect to an orthonormal base
versus

$$\mathbf{D}:=\mathbf{F}-\frac{1}{n}(\text{tr}\,\mathbf{F}\mathbf{G}^{-1})\mathbf{G}\text{, } \textit{in general}$$

measures the deviation of [f_{ij}] = [f_{ji}] from "*trace zero*". [d_{ij}] is accordingly called the *tensor deviator* or *deviatoric tensor*. Another motivation to reduce symmetric multilinear function by their traces is given by "*invariant integration*" which will be outlined as soon as we know how to deal with *active and passive transformations* of geometric objects so far considered.

Example 5-1: Contraction of multilinear functions $\text{tr } f:\mathbf{T}_q^p\to\mathbf{T}_{q-1}^{p-1}$

(i) $\mathbf{T}_0^2\ni f:=\mathbf{e}^{i_1}\otimes\mathbf{e}^{i_2}f_{i_1i_2}\ \forall i_1,i_2\in\{1,2,3\},\ n=3$,

$$\text{tr } f=\sum_{k=1} f_{kk}=f_{11}+f_{22}+f_{33}=\text{tr}\,\mathbf{F}\in\mathbf{T}_0^0.$$

For a (2, 0) tensor-valued function $f\in\mathbf{T}_0^2$ $\text{tr } f$ coincides with the trace of the matrix $\mathbf{F}=[f_{i_1i_2}]\in\mathbb{R}^{3\times3}$, $\dim\mathbf{F}=3\times3$

(ii) $\mathbf{T}_1^2\ni f:=\mathbf{e}^{i_1}\otimes\mathbf{e}^{i_2}\otimes\mathbf{e}_{j_1}f_{i_1i_2}^{j_1}\ \forall i_1,i_2,j_1\in\{1,2,3\},\ n=3$,

$$\text{tr } f(2,1)=\sum_{k=1}^{3}\mathbf{e}^{i_1}f_{i_1k}^{k}=\mathbf{e}^1(f_{11}^1+f_{12}^1+f_{13}^3)+\mathbf{e}^2(f_{21}^1+f_{22}^2+f_{23}^3)+$$
$$+\mathbf{e}^3(f_{31}^1+f_{32}^2+f_{33}^3)\ni\mathbf{T}_0^1=\mathbb{X}^*.$$

For a (2, 1) tensor-valued function $f\in\mathbf{T}_1^2$ $\text{tr } f$ (2, 1) coincides with a vector, whose coordinates are generated by $\Sigma_k f_{i_1}^k$.

(iii) $\mathbf{T}_0^2 \ni f := \mathbf{b}^{i_1} \otimes \mathbf{b}^{i_2} f_{i_1 i_2} \ \forall i_1, i_2 \in \{1,2,3\},\ n = 2$,

$$\operatorname{tr} f = \sum_{k,\ell} f_{k\ell} g^{k\ell} = f_{11} g^{11} + f_{12} g^{21} + f_{21} g^{12} + f_{22} g^{22} = \operatorname{tr} \mathbf{F}\mathbf{G}^{-1} \in \mathbf{T}_0^0.$$

For a (2, 0) tensor-valued function $f \in \mathbf{T}_0^2$ represented in a general coordinate base $\{\mathbf{b}^1, \mathbf{b}^2\}$ tr f coincides with the trace of the product $\mathbf{F}\mathbf{G}^{-1}$, $\mathbf{F} = [f_{k\ell}] \in \mathbb{R}^{2\times 2}$, $\mathbf{G}^{-1} = [g_{k\ell}] \in \mathbb{R}^{2\times 2}$, $\dim \mathbf{F} = \dim \mathbf{G} = 2 \times 2$.

♣

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