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Mixed Integer Observation Models, GPS Decorrelation and Integer Programming

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Preface to the Printing of These Two Papers on GPS

This report consists of the unaltered versions of the two papers written in 1994 and 1995 by P. Xu, E. Cannon and G. Lachapelle. The first paper was presented at the IUGG95, Boulder, Colorado, July 2–14, 1995; the second paper was submitted to a geodetic journal but, due to reasons, was withdrawn by the authors after almost 24 months of review. Although some years have passed, and the papers have been sent to those who requested for a copy from time to time, the new results in these papers have not been known by quite many geodesists working on GPS and are still new today to them. Thus on the invitation of Prof. Erik W. Grafarend, we are pleased to see them in printing in the series of technical report of his Geodetic Institute. Thank you very much, Erik, for suggesting publishing the papers this way.

Mixed integer programming for the resolution of GPS carrier phase ambiguities

Abstract
Ambiguity resolution of GPS carrier phase observables is crucial in high precision geodetic positioning and navigation applications. It consists of two aspects: estimating the integer ambiguities in the mixed integer observation model and examining whether they are sufficiently accurate to be fixed as known nonrandom integers. We shall discuss the first point in this paper from the point of view of integer programming. A one-step nonexact approach is proposed by employing minimum diagonal pivoting Gaussian decompositions, which may be thought of as an improvement of the simple rounding-off method, since the weights and correlations of the floating-estimated ambiguities are fully taken into account. The second approach is to reformulate the mixed integer least squares problem into the standard **0-1** *linear integer programming model, which can then be solved by using, for instance, the practically robust and efficient simplex algorithm for linear integer programming. It is exact, if proper bounds for the ambiguities are given. Theoretical results on decorrelation by unimodular transformation are given in the form of a theorem.*

1 Introduction

Three types of observables may be derived from tracking GPS satellites: pseudorange (code) measurements, raw Doppler shifts (or equivalently range rates) and carrier phases. They are used at different levels of accuracy for different purposes of applications (see e.g Wells et al. 1986; Leick 1990; Hofmann-Wellenhof et al. 1992; Seeber 1993; Melbourne 1985). The carrier phase measurements, together with the accurate code observables (if available), have been dominating in high precision geodetic positioning and navigation applications. The mathematical model can symbolically be written below

$$\mathbf{R} = \mathbf{f}_R(\mathbf{X}) + \mathbf{B}_R\boldsymbol{\lambda} + \varepsilon_R \tag{1a}$$

$$\boldsymbol{\Phi} = \mathbf{f}_\Phi(\mathbf{X}) + \mathbf{B}_\Phi\boldsymbol{\lambda} + \mathbf{B}_Z\mathbf{Z} + \varepsilon_\Phi. \tag{1b}$$

Here $\mathbf{R}$ and $\boldsymbol{\Phi}$ are respectively the observables of pseudoranges and carrier phases, ε_R and ε_Φ are the random errors of the observables, $\mathbf{X}$ is the coordinate vector to be estimated, and $\mathbf{f}_R(.)$ and $\mathbf{f}_\Phi(.)$ are nonlinear functionals of $\mathbf{X}$. $\mathbf{B}_R$, $\mathbf{B}_\Phi$ and $\mathbf{B}_Z$ are the coefficient matrices. $\boldsymbol{\lambda}$ is the vector of nuisance parameters such as the synchronization errors of receiver and satellite clocks and ionospheric corrections. If overparametrization occurs to $\boldsymbol{\lambda}$, it is generally not estimable (Wells et al. 1987). Thus we shall assume that proper reparametrization has been made by, for instance, choosing proper datum parameters (Wells et al. 1987) or using differencing and nuisance parameter elimination techniques (see e.g. Goad 1985; Schaffrin & Grafarend 1986), to ensure that the remaining nuisance parameters are estimable. $\mathbf{Z}$ is the vector of integral ambiguities inhered in the carrier phase observables.

Accurate and reliable resolution of the integral ambiguity vector has been playing a crucial role in high precision positioning. There are currently many approximate proposals available to resolve

$\mathbf{Z}$. They may be treated in two categories: simple (sequential) rounding-off of a real number to its nearest integer with and/or without using constraint criteria (Blewitt 1989; Talbot 1991; Hwang 1991; Seeber 1993; Hofmann-Wellenhof et al. 1992), and searching methods by employing the information on the prior statistics and geometry (nonlinear functionals and design matrices) of the observables (Counselman et al. 1981; Remondi 1990, 1991; Frei & Beutler 1990; Mader 1990; Mervart et al. 1994). Betti, Crespi & Sansò (1993) recently proposed a Bayesian approach to resolution of ambiguity. Chen & Lachapelle (1994) proposed a fast ambiguity search filtering approach to reducing the number of possible candidates in the searching area. It may be worth noting that the fast rapid ambiguity resolution method proposed by Frei & Beutler seems to have enjoyed its wide approval. A key element of the method is the use of some formal statistics to pick up a solution. It may be proved that the statistic used for selecting the candidates of ambiguities is not mathematically rigorous, since the ambiguity-free and ambiguity-fixed solution vectors are both derived by using the same set of carrier phase observations. The method seems quite successful in practice, however.

Recent progress in resolving the integral ambiguity vector has been made by Teunissen (1994). His approach consists of three steps: (1) decorrelation of the floating-estimated ambiguities by Gaussian transformation, which may be said to characterize the novelty of the new approach, (2) searching for the solution to the transformed integer least squares problem within a superellipsoid corresponding to a certain level of confidence, and (3) back-substituting the solution just derived for the ambiguity vector in the original model. The success of the approach will depend, to a great extent, on the first two steps. Testing results of the approach can be found in Teunissen (1994) and de Jonge & Tiberius (1994). Decorrelation techniques may be also well suited to explain an important finding by Melbourne (1985), that the widelane ambiguity is easier to solve, based on the *one epoch* dual frequency carrier phase and code-derived pseudorange model.

The purpose of this paper is to further study the GPS ambiguity resolution as a mixed integer least squares (LS) mathematical programming problem. Unimodular integer transformation is used to statistically decorrelate the floating-estimated ambiguities, which summarizes the first two conditions of transformation proposed by Teunissen (1994). Two methods for solving the transformed integer LS problem are then proposed. The first one is to decompose the transformed positive definite matrix into a lower and an upper triangle by choosing the minimum diagonal elements. In this way, we are sure that a wrongly selected ambiguity will be penalized. No iterations are required, thus it should improve the sum of square of the residuals derived by rounding the transformed real values to their nearest integers. The second one is to reformulate the transformed integer LS problem to a quadratic **0-1** nonlinear programming, and then further to a **0-1** linear integer programming. Thus simplex algorithms can be employed to efficiently solve the linear integer programming problem, with which one need not test every point in the feasible solution set.

2 Integer and mixed integer least squares models

In the application of the GPS system to high precision positioning and navigation, the GPS satellites have been treated as space targets with known positions, unless the determination of the satellite orbits is of interest. In this paper, we assume that the coordinates of the satellites are given, which can be computed, for instance, from the (precision) ephemerides. Furthermore, given a set of approximate coordinates of the stations, we can linearize the observation equations (1a) and (1b) as

$$\mathbf{y}_R = \mathbf{A}_R \Delta\mathbf{X} + \mathbf{B}_R \Delta\boldsymbol{\lambda} + \varepsilon_R \tag{2a}$$

$$\mathbf{y}_\Phi = \mathbf{A}_\Phi \Delta\mathbf{X} + \mathbf{B}_\Phi \Delta\boldsymbol{\lambda} + \mathbf{B}_Z \Delta\mathbf{Z} + \varepsilon_\Phi \tag{2b}$$

where

$$\mathbf{y}_R = \mathbf{R} - \mathbf{f}_R(\mathbf{X}_0) - \mathbf{B}_R \boldsymbol{\lambda}_0 \tag{3a}$$

$$\mathbf{y}_\Phi = \boldsymbol{\Phi} - \mathbf{f}_\Phi(\mathbf{X}_0) - \mathbf{B}_\Phi \boldsymbol{\lambda}_0 - \mathbf{B}_Z \mathbf{Z}_0 \tag{3b}$$

$$\Delta\mathbf{X} = \mathbf{X} - \mathbf{X}_0; \quad \Delta\boldsymbol{\lambda} = \boldsymbol{\lambda} - \boldsymbol{\lambda}_0 \tag{3c}$$

$$\Delta\mathbf{Z} = \mathbf{Z} - \mathbf{Z}_0. \tag{3d}$$

$\mathbf{X}_0$ and $\boldsymbol{\lambda}_0$ are the approximate values of $\mathbf{X}$ and $\boldsymbol{\lambda}$, respectively. $\mathbf{Z}_0$ are integer approximate values of $\mathbf{Z}$, and thus $\Delta\mathbf{Z}$ remain integral.

Rewriting the linearized observation equations (2a) and (2b) in matrix form, together with the statistical information on the observables, we have

$$\begin{bmatrix} \mathbf{y}_R \\ \mathbf{y}_\Phi \end{bmatrix} = \begin{bmatrix} \mathbf{A}_R \\ \mathbf{A}_\Phi \end{bmatrix} \Delta\mathbf{X} + \begin{bmatrix} \mathbf{B}_R \\ \mathbf{B}_\Phi \end{bmatrix} \Delta\boldsymbol{\lambda} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_Z \end{bmatrix} \Delta\mathbf{Z} + \begin{bmatrix} \varepsilon_R \\ \varepsilon_\Phi \end{bmatrix} \tag{4a}$$

$$D\begin{bmatrix} \mathbf{y}_R \\ \mathbf{y}_\Phi \end{bmatrix} = \begin{bmatrix} \mathbf{P}_R & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_\Phi \end{bmatrix}^{-1} \sigma^2. \tag{4b}$$

Here $\mathbf{P}_R$ and $\mathbf{P}_\Phi$ are respectively the weight matrices of the observables $\mathbf{y}_R$ and $\mathbf{y}_\Phi$, σ^2 is the scalar variance component.

Since the main interest of this paper is to discuss the mixed integer LS problem, we do not need to discriminate between the position unknowns $\mathbf{X}$ and the nuisance parameters $\boldsymbol{\lambda}$. Without loss of generality, therefore, we can simplify the model (4) as the following standard mixed real-integer (or simply integer in the rest of the paper) observation equations,

$$\mathbf{y} = \mathbf{A}\boldsymbol{\beta} + \mathbf{B}\mathbf{z} + \varepsilon \tag{5a}$$

$$D(\mathbf{y}) = \mathbf{P}^{-1}\sigma^2 \tag{5b}$$

where

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_R \\ \mathbf{y}_\Phi \end{bmatrix}; \quad \varepsilon = \begin{bmatrix} \varepsilon_R \\ \varepsilon_\Phi \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_R & \mathbf{B}_R \\ \mathbf{A}_\Phi & \mathbf{B}_\Phi \end{bmatrix}; \quad \beta = \begin{bmatrix} \Delta\mathbf{X} \\ \Delta\boldsymbol{\lambda} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_Z \end{bmatrix}; \quad \mathbf{z} = \Delta\mathbf{Z}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_R & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_\Phi \end{bmatrix}.$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ are full of column rank, respectively.

Applying the least squares criterion to (5), we have

$$min: \quad F = (\mathbf{y} - \mathbf{A}\boldsymbol{\beta} - \mathbf{B}\mathbf{z})^T\mathbf{P}(\mathbf{y} - \mathbf{A}\boldsymbol{\beta} - \mathbf{B}\mathbf{z}), \tag{6}$$

which is the mixed integer LS problem. (6) was also called the constrained LS problem by Teunissen (1994). Since the variables $\mathbf{z}$ are discrete, we cannot use the conventional method by differentiating the objective function F with respect to the variables $\boldsymbol{\beta}$ and $\mathbf{z}$ in order to form the normal equation and then solve for them. Instead, however, we differentiate F with respect to $\boldsymbol{\beta}$ and let it equal zero, leading to

$$\frac{\partial F}{\partial \boldsymbol{\beta}} = -2\mathbf{A}^T\mathbf{P}(\mathbf{y} - \mathbf{A}\boldsymbol{\beta} - \mathbf{B}\mathbf{z}) = \mathbf{0}$$

or

$$\mathbf{A}^T\mathbf{P}\mathbf{A}\boldsymbol{\beta} = \mathbf{A}^T\mathbf{P}(\mathbf{y} - \mathbf{B}\mathbf{z}).$$

Hence

$$\boldsymbol{\beta} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P}(\mathbf{y} - \mathbf{B}\mathbf{z}). \tag{7}$$

Substituting (7) into (5) and rearranging it yield

$$\mathbf{y}_1 = \mathbf{QPBz} + \varepsilon_1 \tag{8a}$$

$$D[\mathbf{y}_1] = [\mathbf{P}^{-1} - \mathbf{A}(\mathbf{A}^T\mathbf{PA})^{-1}\mathbf{A}^T]\sigma^2 = \mathbf{Q}\sigma^2 \tag{8b}$$

where

$$\mathbf{y}_1 = [\mathbf{I} - \mathbf{A}(\mathbf{A}^T\mathbf{PA})^{-1}\mathbf{A}^T\mathbf{P}]\mathbf{y} = \mathbf{QPy}$$
$$\mathbf{Q} = \mathbf{P}^{-1} - \mathbf{A}(\mathbf{A}^T\mathbf{PA})^{-1}\mathbf{A}^T.$$

Applying the LS method to (8), we have

$$\begin{aligned} min: \ F_1 &= (\mathbf{y}_1 - \mathbf{QPBz})^T\mathbf{Q}^-(\mathbf{y}_1 - \mathbf{QPBz}) \\ &= (\mathbf{y} - \mathbf{Bz})^T\mathbf{PQQ}^-\mathbf{QP}(\mathbf{y} - \mathbf{Bz}) \\ &= (\mathbf{y} - \mathbf{Bz})^T\mathbf{PQP}(\mathbf{y} - \mathbf{Bz}) \\ &= \mathbf{y}^T\mathbf{PQPy} - 2\mathbf{y}^T\mathbf{PQPBz} + \mathbf{z}^T\mathbf{B}^T\mathbf{PQPBz}. \end{aligned} \tag{9}$$

The objective function F_1 can further be rewritten as

$$min: \quad F_1 = (\mathbf{z} - \hat{\mathbf{z}})^T\mathbf{H}(\mathbf{z} - \hat{\mathbf{z}}) + \mathbf{y}^T\mathbf{PQ}[\mathbf{Q}^- - \mathbf{PBH}^{-1}\mathbf{B}^T\mathbf{P}]\mathbf{QPy} \tag{10}$$

where

$$\hat{\mathbf{z}} = \mathbf{H}^{-1}\mathbf{B}^T\mathbf{PQPy}$$
$$\mathbf{H} = (\mathbf{B}^T\mathbf{PQPB}).$$

Here $\hat{\mathbf{z}}$ can readily be proved to be the floating LS estimate of the ambiguity vector $\Delta\mathbf{Z}$ with covariance matrix $\mathbf{H}^{-1}\sigma^2$. Since $\mathbf{y}^T\mathbf{PQ}[\mathbf{Q}^- - \mathbf{PBH}^{-1}\mathbf{B}^T\mathbf{P}]\mathbf{QPy}$ is constant, the objective function (10) is equivalent to (Teunissen 1994; de Jonge & Tiberius 1994)

$$min: \quad F_2 = (\mathbf{z} - \hat{\mathbf{z}})^T\mathbf{H}(\mathbf{z} - \hat{\mathbf{z}}), \tag{11}$$

which is the standard integer LS problem.

It is now clear that the solution to the original mixed integer LS problem (6) depends solely on that of the standard integer LS problem (11). Denote the integer solution of $\mathbf{z}$ to (11) by $\hat{\mathbf{z}}^{IN}$. Substituting it into (7), we can then obtain the LS estimates of the real parameters $\boldsymbol{\beta}$ without much effort.

3 Unimodular transformation

In resolution of GPS carrier phase ambiguities, one of the most difficult points is to handle strong correlation of the matrix $\mathbf{H}$. Searching for an acceptable (and/or hopefully optimal) solution of $\mathbf{z}$ is arduous, if it is solely based on the strong correlation matrix $\mathbf{H}$, since testing a large number of combinations would have to be done. Roughly speaking, the total number of combinations required is computed by $\prod n_i$, where n_i is the number of integer points on an interval of line for the ith ambiguity, centred at the $\hat{z}_i$ and corresponding to a significance level (see e.g Frei & Beutler 1990). However, if the matrix $\mathbf{H}$ is diagonal, one can simply round the floating values $\hat{\mathbf{z}}$ off to the nearest integers, which are the integer solution of $\mathbf{z}$. Therefore, an idea would emerge naturally, that one works with a decorrelated weight matrix instead of $\mathbf{H}$.

Such a technique was proposed recently by Teunissen (1994) (see also de Jonge & Tiberius 1994). His basic idea is to transform the "observables" $\hat{\mathbf{z}}$ by $\mathbf{G}$ into the new ones $\hat{\mathbf{z}}_1 = (\mathbf{G}^T\hat{\mathbf{z}})$, and then work with the LS integer problem

$$min: \quad F_3 = (\mathbf{z}_1 - \hat{\mathbf{z}}_1)^T\mathbf{H}_1(\mathbf{z}_1 - \hat{\mathbf{z}}_1). \tag{12}$$

Here $\mathbf{H} = \mathbf{G}\mathbf{H}_1\mathbf{G}^T$. The transformation matrix $\mathbf{G}$ has to satisfy the following three conditions: (1) integer elements; (2) volume preservation; and (3) decorrelation of $\mathbf{H}$ into $\mathbf{H}_1$. More details can be found in Teunissen (1994).

Before proceeding, we shall define the unimodular matrix (see e.g. Nemhauser & Wolsey 1988).

Definition 1. A square matrix $\mathbf{G}$ is said to be *unimodular* if it is integral and if the absolute value of its determinant is equal to unity, *i.e.* $|det(\mathbf{G})| = 1$.

The inverse of a unimodular matrix is also unimodular, since $|det(\mathbf{G}^{-1})| = 1/|det(\mathbf{G})| = 1$, and because

$$\mathbf{G}^{-1} = \bar{\mathbf{G}}/det(\mathbf{G}) = \pm\bar{\mathbf{G}}.$$

Here $\bar{\mathbf{G}}$ is the *adjoint matrix* of $\mathbf{G}$, whose elements are derived only by using the operations of integer multiplication, substraction and addition, and thus integer. The sign before $\bar{\mathbf{G}}$ depends on the determinant of $\mathbf{G}$. The second property of unimodular matrices is that the product of two unimodular matrices is unimodular. It is also clear that any unimodular transformation of an integer vector is an integer vector, too.

By employing the concept of the unimodular matrix, we can summarize the first two conditions suggested by Teunissen (1994) by stating that the transformation $\mathbf{G}$ is unimodular. It should be noted that there was a misunderstanding of Teunissen's second condition of volume preservation. Volume preservation does not imply the preservation of the number of grid points. A simple example is that a unit circle centred at the origin has five grid points, while an ellipse of the same center with major axis 1.5 and minor axis 2/3 encloses only three grid points.

Integer Gaussian decomposition was employed by Teunissen (1994), that indeed decorrelates the matrix $\mathbf{H}$. What now seems to be done is to mathematically prove that we can always decorrelate the matrix $\mathbf{H}$ by using a finite number of unimodular transformations to the extent that the correlation coefficient of any two random variables is always less than or equal to 1/2. In order to do so, we need the following lemma on the inequality of matrix determinant.

Lemma 1: For any positive definite matrix $\mathbf{A}$, the following inequality

$$det(\mathbf{A}) \leq \prod a_{ii} \tag{13}$$

holds true. Here a_{ii} are the diagonal elements of $\mathbf{A}$.

Proof. A positive definite matrix $\mathbf{A}$ can be written by Choleski decomposition as

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T$$

where $l_{ii} = (a_{ii} - \sum_{j=1}^{i-1} l_{ij}^2)^{1/2} > 0$. Thus we have

$$\begin{aligned} det(\mathbf{A}) &= \prod l_{ii}^2 \\ &= \prod (a_{ii} - \sum_{j=1}^{i-1} l_{ij}^2) \\ &\leq \prod a_{ii}, \end{aligned}$$

since $\sum_{j=1}^{i-1} l_{ij}^2 \geq 0$. □

Theorem 1: For any positive definite matrix $\mathbf{A}$, there exists a unimodular matrix $\mathbf{G}$ such that

$$\mathbf{A} = \mathbf{G}\mathbf{H}\mathbf{G}^T. \tag{14}$$

Here $\mathbf{H}$ is positive definite, too, and satisfies

$$|h_{ij}| \leq \frac{1}{2}\, min(h_{ii},\ h_{jj}) \quad \forall\, i, j \ \ \&\ \ i \neq j. \tag{15}$$

Proof. Suppose, without loss of generality, that for any three elements a_{ii}, a_{jj} and a_{ij} of the positive definite matrix $\mathbf{A}$, we have $|a_{ij}|/min(a_{ii}, a_{jj}) > 1/2$. Then construct the unimodular matrix

$$\mathbf{G}_1 = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \vdots & \ddots & & \\ & & -[a_{ij}/a_{ii}]_{in} & \cdots & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \tag{16a}$$

if $a_{ii} \leq a_{jj}$, or

$$\mathbf{G}_1 = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & \cdots & -[a_{ij}/a_{jj}]_{in} & & \\ & & & \ddots & \vdots & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \tag{16b}$$

if $a_{jj} < a_{ii}$. Here $[\ x\ ]_{in}$ is the operation to round the floating number x to its nearest integer.

Upon left- and right-multiplying $\mathbf{A}$ by the unimodular matrix $\mathbf{G}_1$ and its transpose respectively, the larger diagonal element is then reduced to

$$max(a_{ii}, a_{jj}) - 2[a_{ij}/a_{min}]_{in} a_{ij} + a_{min}[a_{ij}/a_{min}]_{in}^2 \tag{17}$$

where $a_{min} = min(a_{ii}, a_{jj})$. Repeating the same procedure to any pair of diagonal elements, we have

$$\mathbf{A}_n = \mathbf{G}_n ... \mathbf{G}_1 \mathbf{A} \mathbf{G}_1^T ... \mathbf{G}_n^T. \tag{18}$$

Now suppose that we cannot reach the equation (14) and the inequality (15) by employing a finite number of unimodular matrices of the form (16), then we keep applying the same procedure to $\mathbf{A}_n$. By expression (17), it is clearly true that the minimum diagonal element of the reduced matrix, say $\mathbf{A}_m$ now, has no lower bound. It means that the minimum element can be arbitrarily small, which further implies by Lemma 1 that

$$det(\mathbf{A}_m) \leq \prod a_{ii}^m < const, \tag{19}$$

where a_{ii}^m are the diagonal elements of $\mathbf{A}_m$, *const* is any positive constant. Since unimodular transformation does preserve the determinant, we have $det(\mathbf{A}_m) = det(\mathbf{A})$ — a finite constant, which clearly contradicts (19). Therefore, we must be able to reach the condition (15). On the other hand, all the transformation matrices involved are unimodular, their product is unimodular, too. Denoting the final reduced matrix by $\mathbf{H}$, which satisfies the condition (15), and the product of all the unimodular matrices by $\mathbf{G}_t$, we have

$$\mathbf{H} = \mathbf{G}_t \mathbf{A} \mathbf{G}_t^T \tag{20}$$

or

$$\mathbf{A} = \mathbf{G}\mathbf{H}\mathbf{G}^T. \tag{21}$$

Here $\mathbf{G}(= \mathbf{G}_t^{-1})$ is unimodular. The proof that the matrix $\mathbf{H}$ is positive definite is trivial. □

4 Two approaches to the integer LS problem

The integer LS problem is simply an integer quadratic programming issue. One can use any advanced integer programming algorithm (Parker & Rardin 1988) to solve this problem. Essentially, no bounds for the integer unknowns are required and no statistical techniques needed to reduce the number of possible candidates. More on these aspects and proper validation criteria for fixing the carrier phase ambiguities will be presented in a future paper.

Though the techniques to be presented below require no decorrelation as an assumption, and consider that the original and the transformed LS integer problems are of the same form, the following discussion will be based on the transformed model, without loss of generality. After the weight matrix of the floating-estimated ambiguity vector is decorrelated, one can either simply round the transformed floating numbers off to their nearest integers, or employ searching techniques to find the "optimal" solution within a superellipsoid under a certain level of confidence (Teunissen 1994). In what follows, we shall develop two approaches to resolve the ambiguities of the transformed integer LS problem.

4.1 A one-step nonexact approach by minimum diagonal pivoting Gaussian decomposition

Instead of directly applying the simple rounding-off method to (12), which ignores any correlation information on the floating-estimated ambiguities, we propose an alternative one-step approach, based on the weights and correlations of the transformed ambiguities. The basic idea is to resolve the integer ambiguities according to their weights and correlations. As long as some of ambiguities are resolved, their correlations with other unfixed floating ambiguities are employed and the next ambiguity corresponding to the large weight is to be determined.

In order to realize the above procedure, we have to decompose the positive definite matrix $\mathbf{H}_1$ carefully. Here we employ Gaussian decomposition by selecting the minimum diagonal element. The decomposition procedure consists of the following steps:

- Selecting the minimum element among all the undecomposed diagonal elements;
- Exchanging the rows and the columns;
- Performing Gaussian decomposition;
- Replacing the square root of the decomposed element $h'_{1(ii)}$ at the corresponding position of the factor matrix $\mathbf{L}$; If the decomposition is not completed, then go to the first step. Otherwise, the decomposition is finished.

In mathematical language, we can express the matrix $\mathbf{H}_1$ as

$$\mathbf{H}_1 = \mathbf{P}_h \mathbf{L}\mathbf{L}^T \mathbf{P}_h^T \tag{22}$$

where $\mathbf{P}_h$ is the permutation matrix which represents the exchange of the rows and columns during the decomposition. A significant characteristic of this decomposition is to keep the diagonal elements of the lower triangular matrix $\mathbf{L}$ in the increasing order as far as possible.

Inserting $\mathbf{H}_1$ in (22) into (12), we have the objective function

$$\begin{aligned} min: \quad F_3 &= (\mathbf{z}_1 - \hat{\mathbf{z}}_1)^T \mathbf{P}_h \mathbf{L}\mathbf{L}^T \mathbf{P}_h^T (\mathbf{z}_1 - \hat{\mathbf{z}}_1) \\ &= (\mathbf{z}_2 - \hat{\mathbf{z}}_2)^T \mathbf{L}\mathbf{L}^T (\mathbf{z}_2 - \hat{\mathbf{z}}_2) \end{aligned} \tag{23}$$

where

$$\mathbf{z}_2 = \mathbf{P}_h^T \mathbf{z}_1; \quad \hat{\mathbf{z}}_2 = \mathbf{P}_h^T \hat{\mathbf{z}}_1. \tag{24}$$

Since the factor matrix $\mathbf{L}$ is lower triangular, we can rewrite (23) as

$$min: \quad F_4 = \sum_{i=1}^{t_z} [\sum_{j=i}^{t_z} l_{ji}(z_{2(j)} - \hat{z_{2(j)}})\ ]^2. \tag{25}$$

Here t_z is the dimension of the ambiguity vector $\mathbf{z}$ (or $\mathbf{z}_2$). The solution to the objective function F_2 can now be derived by minimizing

$$|\sum_{j=i}^{t_z} l_{ji}(z_{2(j)} - \hat{z}_{2(j)})|, \quad \forall\ i. \tag{26}$$

Hence the one-step nonexact integer ambiguity solution is immediate

$$\hat{z}_{2(i)}^{IN} = \left[\frac{l_{ii}\hat{z}_{2(i)} - \sum_{j=i+1}^{t_z} l_{ji}(\hat{z}_{2(j)}^{IN} - \hat{z}_{2(j)})}{l_{ii}}\right]_{in} \tag{27}$$

for all i.

By back substituting the integer solution $\hat{\mathbf{z}}_2^{IN} = (\hat{z}_{2(1)}^{IN}, \hat{z}_{2(2)}^{IN}, ..., \hat{z}_{2(t_z)}^{IN})^T$, we have the final solution of the integer ambiguities $\mathbf{z}$, which is denoted by $\hat{\mathbf{z}}^{IN}$,

$$\hat{\mathbf{z}}^{IN} = \mathbf{G}^{-T}\mathbf{P}_h\ \hat{\mathbf{z}}_2^{IN}. \tag{28}$$

4.2 0-1 quadratic integer programming

An obvious aim of applying the decorrelation technique to the original integer LS problem is the alleviation of the computational burden for finding the optimal ambiguity solution. When it is translated into the case of searching techniques, we expect that the total number of candidate grid points to be tested should be significantly reduced. Suppose that for the transformed integer LS problem (12) ($\mathbf{H}_1$ satisfies the conditions of Theorem 1), we have to search for the optimal integer ambiguity resolution within the hard bounds

$$m_i^0 \leq z_{1(i)} \leq m_i^1, \quad \forall\ i \tag{29}$$

or in another form,

$$z_{1(i)} \in [m_{1i}(= m_i^0),\ m_{2i},\ ...,\ m_{1s_i}(= m_i^1)]. \tag{30}$$

Here $z_{1(i)}$ is the ith integer component of the integer vector $\mathbf{z}_1$, m_{1i}, m_{2i}, .., and m_{1s_i} are the contiguous integers — the candidate points of $z_{1(i)}$ with the lower integer bound m_i^0 and the upper integer bound m_i^1. Thus our mixed integer LS problem has been reduced to a quadratic integer programming problem with simple integer constraints.

In what follows we shall further reformulate it by a **0-1** quadratic integer programming model. It has been shown by Parker & Rardin (1988) that the integer variable $z_{1(i)}$ can be represented with r_i **0-1** variables, *i.e.*

$$z_{1(i)} = m_i^0 + \sum_{j=0}^{r_i-1} 2^j\ b_{i(j)}, \quad \forall\ i \tag{31}$$

where $b_{i(j)}$ are **0-1** integer (binary) variables, $r_i = [log_2(m_i^1 - m_i^0)]_s + 1$, and $[\ .\ ]_s$ stands for the integer not larger than the positive number in brackets.

Rewriting all the integer variables $z_{1(i)}$ in matrix form, we have

$$\mathbf{z}_1 = \mathbf{m}^0 + \mathbf{A}_1\mathbf{b} \tag{32}$$

where the matrix $\mathbf{A}_1$ is integral with elements 2^k,

$$\mathbf{m}^0 = (m_1^0,\ m_2^0,\ ...,\ m_{t_z}^0)^T$$

$$\mathbf{b} = (\mathbf{b}_1^T,\ \mathbf{b}_2^T,\ ...,\ \mathbf{b}_{t_z}^T)^T$$

$$\mathbf{b}_i = (b_{i(0)},\ b_{i(1)},\ ...,\ b_{i(r_i-1)})^T.$$

Furtheron, inserting (32) into the objective function (12) yields

$$min: \quad F_3 = (\mathbf{A}_1\mathbf{b} + \mathbf{m}^0 - \hat{\mathbf{z}}_1)^T\mathbf{H}_1(\mathbf{A}_1\mathbf{b} + \mathbf{m}^0 - \hat{\mathbf{z}}_1) \tag{33}$$

subject to $b_k = 0$ or 1 for all k.

The objective function (33) is equivalent to

$$\begin{aligned} min: \quad F_3 &= (\mathbf{m}^0 - \hat{\mathbf{z}}_1)^T\mathbf{H}_1(\mathbf{m}^0 - \hat{\mathbf{z}}_1) + 2(\mathbf{m}^0 - \hat{\mathbf{z}}_1)^T\mathbf{H}_1\mathbf{A}_1\mathbf{b} \\ &\quad + \mathbf{b}^T\mathbf{A}_1^T\mathbf{H}_1\mathbf{A}_1\mathbf{b}. \end{aligned} \tag{34}$$

4.3 0-1 linear integer programming

In this subsection, we shall further reformulate the **0-1** quadratic programming (34) into a **0-1** linear integer programming problem by using the linearization technique. The basic idea of the linearization technique is to introduce a new variable to replace the nonzero quadratic term b_ib_j. Thus the **0-1** quadratic programming problem becomes linear. Since the new variables are obviously binary, all the variables in the linear programming model to be reformulated below are binary, too.

Denoting

$$v_k = b_ib_j, \quad k = (i-1)i/2 + j, \quad i \geq j$$

and taking the following relations

$$b_i^2 = b_i$$

into account, we have

$$min: \quad F_4 = (\mathbf{m}^0 - \hat{\mathbf{z}}_1)^T\mathbf{H}_1(\mathbf{m}^0 - \hat{\mathbf{z}}_1) + \sum_{i=1}^{t_v} c_iv_i \tag{35a}$$

subject to the following constraints,

$$v_i = 0 \vee 1 \tag{35b}$$

$$v_k \geq v_{ki} + v_{kj} - 1 \tag{35c}$$

$$v_k \leq v_{ki} \tag{35d}$$

$$v_k \leq v_{kj} \tag{35e}$$

$$ki = i(i+1)/2; \quad kj = j(j+1)/2.$$

Here t_v is the dimension of the **0-1** integer vector

$$\mathbf{v} = (v_1, v_2,\ ...,\ v_{t_v})^T.$$

Since the first term in the objective function (35a) is constant, it is equivalent to

$$min: \quad F_4 = \sum_{i=1}^{t_v} c_iv_i \tag{36a}$$

subject to the constraints (35b $\sim$ e). (36) is obviously of the standard form of the **0-1** linear integer programming. It can be solved by using any standard algorithms for **0-1** linear programming (Pardalos & Li 1993; Nemhauser & Wolsey 1988; Parker & Rardin 1988; The People University of China 1987). However, the algorithm aspects for the program (36) will not be discussed here.

5 Concluding remarks

GPS carrier phase and pseudorange observables are essentially a nonlinear mixed integer observation model. If the GPS satellites are treated as space known targets, the model is regular. Given a set of approximate values of the unknown parameters such as the position coordinates and integer ambiguities, the nonlinear model is linearized. Estimating the parameters in the linearized mixed integer model is equivalent to solving a mixed integer LS problem (if the LS principle is employed), which can be further reduced into a standard integer LS programming.

It has been recognized that one of the difficulties in correctly estimating the integer ambiguities is due to the correlations of the floating-estimated ambiguities. A decorrelation technique has been proposed by Teunissen (1994), based on Gaussian decomposition. We have further proved mathematically that there exists a unimodular matrix such that (14) and (15) hold true, which may be thought of as a theoretical summary (and extension) of some of the results in Teunissen (1994).

Two approaches are then proposed to solve the standard linear integer LS problem (12) from the point of view of integer programming theory. The first approach is to Gauss-decompose the matrix $\mathbf{H}_1$ by selecting the minimum diagonal elements. In other words, we are estimating the integer ambiguities according to the magnitudes of the weights of the floating-estimated ambiguities and their correlations (as far as possible). It may be thought to be an improvement of the simple rounding-off method. No iterations are required. It should be noted, however, that this method is one-step nonexact. The extent of approximation should be further investigated. The second approach is to reformulate the mixed integer LS problem into a **0-1** linear integer programming model. Thus any standard algorithms for linear integer programming problems can be employed. The method will result in the exact integer solution of the ambiguities to the original mixed integer problem, if proper bounds for the integer unknowns in the transformed model (12) are given. Testing of the techniques with real data is under way.

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GPS ambiguity resolution by integer quadratic programming

Abstract This paper is to further investigate the applicability of integer programming to resolve GPS ambiguities with real data and thus also complements earlier theoretical results on this topic. The purposes of this study are twofold. One is to discuss the GPS ambiguity problem from the point of view of integer programming, and then implement the branch and bound method to quickly solve the integer ambiguities. The advantages of using integer programming include: (1) searching for the solution is smartly carried out automatically so that many non-promising points will not be tested; and (2) a searching bounded box is not required so that the possibility of excluding the correct solution from the pre-selected bounded box is completely avoided. The other is to discuss validation criteria for assessment of the correctness of the ambiguities obtained. A shipborne data set is used to demonstrate how the approach works.

Key words: GPS ambiguity resolution, mixed integer observation models, integer programming.

1 Introduction

High precision geodetic positioning and navigation has been an important aspect of GPS applications. In deformation measurement, it has been demonstrated that GPS can be used to detect (relative) displacements in seismically active areas at the millimetre level (Feigl et al. 1993). Aeroplane landing supported by GPS with the aid of pseudolites has been extensively investigated, especially by a group of scientists at Stanford University, and the results have shown the promising potential to satisfy FAA (Federal Aviation Adminstration) Category III landing requirements (Pervan et al. 1994; Cohen et al. 1995; van Graas et al. 1995). The attitude of a platform can be determined at the level of within one or two arc minutes by a GPS multiantenna system, depending on the separations and configuration of the antennas (Lu 1995; Cannon & Sun 1996). Another application is precise farming, such as soil salinity measurements, combine harvester guidance and fertilier spreading (see *e.g.* Cannon et al. 1994; Lachapelle et al. 1994; Cannon et al. 1997).

High precision applications of GPS depend essentially on the use of carrier phase observables, which are, however, inherent to unknown integer ambiguities. Thus one GPS research interest has been to solve for these integer ambiguities in order to fully exploit the advantages of precise carrier phase observables. There are a number of techniques proposed in the GPS literature. In terms of searching space, these techniques may be classified into two classes: ambiguity-space based searching methods and coordinate-space based searching methods. The latter is characterized by the ambiguity function method, which was originated from Counselman & Gourevitch (1981) and then applied by Remondi (1990) and Mader (1990), among others.

The former received most extensive and intensive investigations. A number of methods have been presented and tested. The common points of ambiguity-space and coordinate-space based techniques may be summarized as follows: (1) Solving the floating ambiguity solution by treating all the integer variables in the mixed integer least squares (LS) problem as simple real numbers; One can then straightforwardly obtain the floating solution by solving the linear normal equations. (2) Setting up lower and upper bounds for the integer ambiguities, by using the statistical information from the floating solution, for instance; Very often, a scaling factor of 3 to 15 is employed (see, e.g. Frei & Beutler 1990; Seeber 1993; Hofmann-wellenhof et al. 1992). Further constraints may be applied as in Abidin (1993). Then one will search over the bounded area for the optimal integer ambiguity solution. (3) Reducing nonpromising candidates within the searching area; In this aspect, Frei & Beutler (1990) suggested using statistical testing to eliminate nonpromising gridding points. Recently, Teunissen proposed to apply a decorrelation technique to the original set of ambiguities in order to derive a better shaped error ellipsoid (Teunissen 1994, 1995, 1996; de Jonge & Tiberius 1994). Based on the concept of conditional variance to some extent, a fast ambiguity search filtering (FASF) algorithm was proposed by Chen (1994) and Chen & Lachapelle (1994) to reduce the number of possible candidates to be tested. Other possible techniques may be inferred from, e.g. Hatch (1990), Hwang (1990), Talbot (1990), Hein & Werner (1995) and Martin-Neira et al. (1995). (4) Validating the integer ambiguity solution obtained after the first three steps (see, *e.g.* Abidin 1993; Frei & Butler 1990; Teunissen 1994); A criterion used most frequently is based on the minimum and second minimum sums of squares of the computed residuals of the carrier phase observables. An empirical constant will dominate the success of this criterion, since the ratio of the second minimum to the minimum is essentially not F-distributed.

Integer programming techniques, aimed at solving an optimization problem with all variables of integer nature, seem to have been almost completely ignored in the resolution of GPS carrier phase ambiguities, though, the ambiguities are of integer nature. Recently, Xu et al. (1995) started investigating this problem. They defined the mixed integer observation model and the mixed integer LS problem from the view point of integer programming, and then further formulated the GPS ambiguity problem into a few integer programming models. The basic idea of solution is different from the current approaches. The second and third steps, as summarized above, have no role to play. No bounds for the ambiguities (even in the form of an ellipsoid) are required and no statistical testing is needed in order to reduce nonpromising gridding points. Similar work was done by Wei & Schwarz (1995), who, together with these authors, are among the first to investigate potential applications of integer programming to geomatics, in particular to GPS ambiguity resolution. Although the work by Wei & Schwarz (1995) and the present paper are based on integer programming to solve the GPS ambiguity problem, they are mainly different in the employment of solution algorithms and validation criteria.

The purpose of this paper is twofold. One is to discuss the GPS ambiguity problem from the point of view of integer programming, and then implement the branch and bound method to quickly solve the integer ambiguities. The other is to discuss validation criteria for assessment of the correctness of the ambiguities obtained. Consider that the ratio of the second minimum and minimum sums of squared residuals has been connected with some empirical constants and, to some extent, can control the correctness of the resolved ambiguities globally, we will propose a simple validation criterion of ambiguity repeatability over time. Section 2 will briefly discuss the mixed GPS integer observation model and the GPS integer quadratic objective. Section 3 will implement the branch and bound method to solve the integer convex quadratic programming model formulated in section 2. Section 4 will focus on validating the solved ambiguities, which follows by section 5 on computation of a real data set.

2 Defining mixed integer observation models and integer LS problems

2.1 The classical (real-valued) observation model and LS method

Given a number of real-valued measurements $\mathbf{y}$ and a real-valued model $f(\mathbf{X})$ ($\mathbf{X} \in \mathbf{R}^n$), and assume that the measurements are related to the model through the (linear or nonlinear) functional relationship:

$$\mathbf{y} = f(\mathbf{X}) + \boldsymbol{\epsilon} \tag{1}$$

Here $\boldsymbol{\epsilon}$ is the random vector, whose first moment is assumed zero and whose second central moment $\mathbf{P}^{-1}\sigma^2$. Equation (1) is the conventional (real-valued) observation model and often is summarized as follows (see, *e.g.* Krakiwsky 1990)

$$\left.\begin{array}{l} \mathbf{y} = f(\mathbf{X}) + \boldsymbol{\epsilon}, \quad E(\mathbf{y}) = f(\mathbf{X}) \\ D(\mathbf{y}) = D(\boldsymbol{\epsilon}) = \mathbf{P}^{-1}\sigma^2 \end{array}\right\} \tag{2}$$

Applying the least squares criterion to (2), we have the minimization problem:

$$\min: \quad [\mathbf{y} - f(\mathbf{X})]^T \mathbf{P} [\mathbf{y} - f(\mathbf{X})], \tag{3}$$

which is actually the classical (real-valued) LS method.

2.2 Mixed integer observation models and integer LS problems

A fundamental characteristic of the classical observation model and LS problem is that both the measured quantities and the parameters to be estimated are real-valued. This set-up is no longer sufficient in the GPS era, since some of the parameters in a model can only take on integer values. As is well known, carrier phase observables are crucial for high precision positioning and navigation. However, an ambiguity of full cycles is inherited in them. As a simplification, one may treat the integer unknowns as real variables and then apply the classical LS method to estimate the quantities of interest. This is unfortunately proved to be not viable for high precision application of GPS signals. Therefore, we have to deal with this new types of observation models of continuous real-valued and discontinuous integer variables.

Given a number of real-valued measurements and a (linear or nonlinear) functional of two types of variables: continuous and discrete, we have the following relationship:

$$\mathbf{y} = f(\mathbf{X}, \mathbf{z}) + \boldsymbol{\epsilon} \tag{4}$$

where $\mathbf{y}$ is a real-valued observation vector, $\mathbf{X}$ is a real-valued parameter vector to be estimated from $\mathbf{y}$, $\mathbf{z}$ is an unknown integer vector, $\boldsymbol{\epsilon}$ is the random vector, $f(.)$ is a functional. In order to complete the description of model (4), we will further assume that the first moment of $\boldsymbol{\epsilon}$ is zero and its second central moment $\mathbf{P}^{-1}\sigma^2$. Thus the mixed integer observation model can now be given below

$$\left.\begin{array}{l} \mathbf{y} = f(\mathbf{X}, \mathbf{z}) + \boldsymbol{\epsilon}, \quad E(\mathbf{y}) = f(\mathbf{X}, \mathbf{z}) \\ D(\mathbf{y}) = D(\boldsymbol{\epsilon}) = \mathbf{P}^{-1}\sigma^2 \end{array}\right\} \tag{5}$$

where $\mathbf{X} \in \boldsymbol{R}^n$, $\mathbf{z} \in \boldsymbol{Z}^m$, and $\boldsymbol{R}^n$ and $\boldsymbol{Z}^m$ are respectively an n-dimensional real-valued space and an m-dimensional integer space.

In terms of mixed integer model (5), the LS minimization objective function becomes

$$\min: \quad [\mathbf{y} - f(\mathbf{X}, \mathbf{z})]^T \mathbf{P} [\mathbf{y} - f(\mathbf{X}, \mathbf{z})], \tag{6}$$

By comparing (5) with (2), it is immediately obvious that the difference between them is in the introduction of integer variables into (5). This is, however, fundamental and invites much complexity to the mixed model (5), computationally and statistically. In the case of linear real-valued LS models, the LS problem is simply equivalent to solving a system of linear equations, and the accuracy measure can be directly derived from the inversion of the normal matrix as a by-product of the solution. There is no easy way, however, to solve the mixed integer LS problem (6) numerically. In order to estimate the integer parameters, one has to resort some sophisticated algorithms in integer programming and the burden of computation increases substantially. On the other hand, statistical aspects of the estimate are no longer obvious and it is even difficult to get a realistic accuracy indicator for the estimate, as is the situation in applying GPS to precise positioning and navigation today. Further research is required.

In what follows, we assume that the functional relationship is linear. Then we will further define a number of linear mixed integer observation models.

Case (A): The linear mixed integer observation model

Assume that the functional relationship in (5) is linear with respect to the variables $\mathbf{X}$ and $\mathbf{z}$. Then (5) can be rewritten as follows:

$$\left.\begin{array}{l}\mathbf{y} = \mathbf{AX} + \mathbf{Bz} + \boldsymbol{\epsilon}, \quad E(\mathbf{y}) = \mathbf{AX} + \mathbf{Bz} \\ D(\mathbf{y}) = D(\boldsymbol{\epsilon}) = \mathbf{P}^{-1}\sigma^2\end{array}\right\} \tag{7}$$

where $\mathbf{A}$ and $\mathbf{B}$ are nonzero design matrices, $\mathbf{X} \in \boldsymbol{R}^n$, $\mathbf{z} \in \boldsymbol{Z}^m$, and $\boldsymbol{R}^n$ and $\boldsymbol{Z}^m$ are respectively an n-dimensional real-valued space and an m-dimensional integer space. The other quantities have been defined in (5). When the LS method is applied to (7), it is correspondingly called the mixed integer LS problem.

Case (B): The linear integer observation model

Let $\mathbf{A} = 0$ in (7). Then we have the linear integer observation model

$$\left.\begin{array}{l}\mathbf{y} = \mathbf{Bz} + \boldsymbol{\epsilon}, \quad E(\mathbf{y}) = \mathbf{Bz} \\ D(\mathbf{y}) = D(\boldsymbol{\epsilon}) = \mathbf{P}^{-1}\sigma^2\end{array}\right\} \tag{8}$$

whose parameters to be estimated are all integral. The corresponding LS minimization is called the linear integer LS problem.

Case (C): The mixed linear $\mathbf{0}-\mathbf{1}$ observation model

Case (C) is different from Case (A) in that the integer variables of Case (C) can take on either one or zero. The model becomes

$$\left.\begin{array}{l}\mathbf{y} = \mathbf{AX} + \mathbf{Bz} + \boldsymbol{\epsilon}, \quad E(\mathbf{y}) = \mathbf{AX} + \mathbf{Bz} \\ D(\mathbf{y}) = D(\boldsymbol{\epsilon}) = \mathbf{P}^{-1}\sigma^2\end{array}\right\}, \tag{9}$$

where $z_i \in \{0,\ 1\}$. The other quantities have been defined as in (7).

Case (D): The linear $\mathbf{0}-\mathbf{1}$ observation model

Assume that all the integer variables in (8) are Boolean (or 0-1) variables. Then we have the following linear $\mathbf{0}-\mathbf{1}$ observation model

$$\left.\begin{array}{l}\mathbf{y} = \mathbf{Bz} + \boldsymbol{\epsilon}, \quad E(\mathbf{y}) = \mathbf{Bz} \\ D(\mathbf{y}) = D(\boldsymbol{\epsilon}) = \mathbf{P}^{-1}\sigma^2\end{array}\right\}, \tag{10}$$

where $z_i \in \{0,\ 1\}$.

The LS problem with respect to this model is a $\mathbf{0}-\mathbf{1}$ quadratic programming model, i.e.

$$\min: \quad (\mathbf{y} - \mathbf{Bz})^T\mathbf{P}(\mathbf{y} - \mathbf{Bz}) \tag{11}$$

Case (E): The linear $\mathbf{0}-\mathbf{1}$ programming model

Since any integer bounded from below and above can be expressed in a finite number of Boolean variables, all the integer-related LS problems can always be transformed into the following linear $\mathbf{0}-\mathbf{1}$ programming problem:

$$\begin{array}{ll}\min: & F = \boldsymbol{a}^T\boldsymbol{b} \\ s.t. & b_i = 0,\ 1. \quad \forall\ i.\end{array} \tag{12}$$

GPS ambiguity parameters are integral. If they are bounded (see *e.g.* Abidin 1993; Teunissen 1996), we can then express GPS integer ambiguities by using Boolean variables. Thus Cases (C), (D) and (E) could also be used to deal with the GPS ambiguity resolution problem (if necessary). More details can be found in Xu et al. (1995).

3 The branch and bound approach to GPS ambiguity resolution

The determination of interested positions and GPS integer ambiguities from carrier phase observables is, in principle, to solve a mixed integer LS problem (1). It has been shown, however, that any mixed GPS integer ambiguity LS problem can be transformed into an integer LS quadratic programming model (see, e.g. Teunissen 1994; Xu et al. 1995). Thus we will confine ourselves to the solution to the GPS integer ambiguity LS problem of model (8), i.e.

$$\min: \quad (\mathbf{y} - \mathbf{Bz})^T\mathbf{P}(\mathbf{y} - \mathbf{Bz}) \tag{13}$$

where $\mathbf{z} \in \boldsymbol{Z}^m$.

There may be several techniques in linear integer programming that could be employed to solve the integer LS problem (13). If the integer variables are all bounded, then one can use the enumeration technique to find the optimal solution to the quadratic programming (13). Since we do not have any such prior information on the integer variables, it is impossible to directly implement it for the solution to (13). One may argue that the matrix $\mathbf{P}$ has been used to bound GPS ambiguities in the form of an ellipsoid (Wuebenna 1991; Abidin 1993; Tennissen 1994). We should like to note: (1) that the use of $\mathbf{P}$ to set up the bounds for $\mathbf{z}$ is actually based on the information in (13) but not on independent prior information; and (2) that one should be careful in using $\mathbf{P}$ to bound $\mathbf{z}$, since there might exist a possibility to single out the correction solution from the bounded box. It can be proven, fortunately, that if the objective function is globally convex, then we can still use the idea of enumeration to solve an unconstrained optimization problem (see *e.g.* Taha 1975). One of such techniques to solve an unconstrained but globally convex optimization problem such as (13) is to use the branch and bound method in integer programming. Since the quadratic objective function is strictly convex, the optimal solution is unique in probability. Although the uniqueness of the solution to the LS integer problem can be guaranteed in probability, its correctness will depend on the probability of the solution and thus has to be checked (or validated) further. In what follows, we will discuss the principle of the branch and bound method and its solution strategy. More details on integer programming can be found in Garfinkel & Nemhauser (1972), Nemhauser & Wolsey (1988), Parker & Rardin (1988), and Taha (1975).

3.1 The branch and bound method for GPS ambiguity resolution

GPS ambiguity resolution can be generalized by the following integer optimization problem:

$$\min: \quad g(\mathbf{z}), \quad \mathbf{z} \in \boldsymbol{Z}^m \tag{14}$$

where $g(.)$ is a convex function, $\mathbf{z}$ is a GPS integer ambiguity vector.

The solution approach to (14) can be solved by branching and bounding, which involves two important steps: computing the lower bound (upper bound in case of maximization) and fathoming. Suppose that the optimization problem (14) has been currently fathomed into j optimization problems with different constraints and arranged in a basic tree enumeration. The problem to be considered now is

$$\min: \quad g(\mathbf{z}), \quad \mathbf{z} \in \boldsymbol{Z}_j^m \tag{15}$$

Then there will be three possibilities that happen to (15): (a). There are no feasible solutions in $\boldsymbol{Z}_j^m$; (b). The objective value at the solution point is unbounded, i.e. $g(\mathbf{z}_j^*) = -\infty$; (c). There is a unique solution that solves (15), which is denoted by $\mathbf{z}_j^*$. In case (a), there is no need to fathom further and the tree branch can be cut off here. Then go to a new problem (or the end of a branch of the enumeration tree) and repeat the procedure described. Case (b) indicates that (15) is unbounded from below.

Now we will further consider case (c). Compare $g(\mathbf{z}_j^*)$ with the current optimal objective value, which is denoted by g_0. If $g(\mathbf{z}_j^*) \geq g_0$, no further fathoming is necessary. Go to a new problem and repeat the procedure as for the jth problem. If $g(\mathbf{z}_j^*) < g_0$, and if $\mathbf{z}_j^*$ is a feasible solution, then replace g_0 by $g(\mathbf{z}_j^*)$, go to a new problem and repeat. Otherwise, we have to fathom $\boldsymbol{Z}_j^m$ into two disconnected subspaces, say $\boldsymbol{Z}_{j1}^m$ and $\boldsymbol{Z}_{j2}^m$. Thus two new branches grow from the current vertex j. Choose one of the ends of the tree branches and repeat the branching and bounding procedure.

The procedure of branching and bounding described above can be summarized algorithmically as follows:

Step 1: (Initialization.) Start with a feasible solution (say $\mathbf{z}_0$) and compute the objective value which is denoted by g_0. Treat all the integer variables as real variables. Then solve the real-valued version of the optimization problem (14). If the solution is of integer nature, go to Step 7; otherwise select an element of the real-valued solution $\mathbf{z}_r$ to branch the original problem into two equivalent problems and put them in the problem list, then go to Step 2.

Step 2: (Solving a problem in the problem list.) Check whether there are any problems in the problem list that were not solved yet. If the answer is negative, go to Step 7; otherwise solve a problem in the list that was not solved yet and go to Step 3.

Step 3: (Fathoming, case a.) If there is no solution to the problem selected in Step 2, delete it from the problem list and go to Step 2. Otherwise go to Step 4.

Step 4: (Fathoming, case b.) If the solution is feasible and if the objective value is equal to $-\infty$, go to Step 7. Otherwise go to Step 5.

Step 5: (Fathoming, case c.) Let the objective value be $g(\mathbf{z}_j^*)$. If $g(\mathbf{z}_j^*) \geq g_0$, no further fathoming is necessary. Thus go to Step 2. Otherwise go to Step 6.

Step 6: (Determining new lower bounds and further fathoming.) Check if the solution is of integer nature. If the answer is positive, replace g_0 and $\mathbf{z}_0$ by $g(\mathbf{z}_j^*)$ and $\mathbf{z}_j^*$, respectively. Go to Step 2. Otherwise select the element of $\mathbf{z}_j^*$ that is not integral to branch and add the two new problems into the problem list, then go to Step 2.

Step 7: (Termination.) If $g_0 = -\infty$, there is no feasible solution to the original optimization problem. If $g_0 > -\infty$, $\mathbf{z}_0$ is the optimal solution and g_0 is the minimum objective value.

3.2 Linear least squares solutions with equality and inequality constraints

When the branch and bound method is employed to solve the GPS integer ambiguity LS problem (13) or (14), an equivalent set of problems with inequality and/or equality constraints are generated, though the original LS objective is unconstrained. Thus solving a linear LS problem with equality and/or inequality constraints plays a role in solving the original problem (13), which is to be discussed in this subsection.

A linear LS problem with equality and inequality constraints can be stated as follows:

$$\begin{array}{ll} \min: & (\mathbf{y}-\mathbf{B}\mathbf{z}_r)^T\mathbf{P}(\mathbf{y}-\mathbf{B}\mathbf{z}_r) \\ s.t. & \mathbf{D}\mathbf{z}_r = \boldsymbol{e} \\ & \mathbf{G}\mathbf{z}_r - \boldsymbol{w} \geq 0, \end{array} \tag{16}$$

where $\mathbf{z}_r \in \boldsymbol{R}^m$, $\mathbf{D}$ and $\mathbf{G}$ are real coefficient matrices, and $\boldsymbol{e}$ and $\boldsymbol{w}$ are constant vectors.

Since for the inequality constraints there exists a positive vector (say $\boldsymbol{h}$) that can make them become equality constraints, we can rewrite (16) as follows:

$$\begin{array}{ll} \min: & (\mathbf{y}-\mathbf{B}\mathbf{z}_r)^T\mathbf{P}(\mathbf{y}-\mathbf{B}\mathbf{z}_r) \\ s.t. & \mathbf{D}\mathbf{z}_r = \boldsymbol{e} \\ & \mathbf{G}\mathbf{z}_r - \boldsymbol{w} - \boldsymbol{h} = 0, \end{array} \tag{17}$$

Define the extended objective by

$$\begin{aligned} f(\mathbf{z}_r, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{h}) &= \frac{1}{2}(\mathbf{y}-\mathbf{B}\mathbf{z}_r)^T\mathbf{P}(\mathbf{y}-\mathbf{B}\mathbf{z}_r) \\ &\quad -\boldsymbol{p}^T(\mathbf{D}\mathbf{z}_r - \boldsymbol{e}) \\ &\quad -\boldsymbol{q}^T(\mathbf{G}\mathbf{z}_r - \boldsymbol{w} - \boldsymbol{h}). \end{aligned} \tag{18}$$

Here we require that $\boldsymbol{h} \geq 0$ and $\boldsymbol{q} \geq 0$ (see, e.g. Golub & Saunders 1970).

Differentiating $f(.)$ with respect to $\mathbf{z}_r$, $\boldsymbol{p}$ and $\boldsymbol{q}$, and then equating them to zero, we get

$$\left.\begin{array}{c} \mathbf{B}^T\mathbf{P}\mathbf{B}\hat{\mathbf{z}}_r - \mathbf{D}^T\boldsymbol{p} - \mathbf{G}^T\boldsymbol{q} = \mathbf{B}^T\mathbf{P}\mathbf{y} \\ \mathbf{D}\mathbf{z}_r = \boldsymbol{e} \\ \mathbf{G}\hat{\mathbf{z}}_r - \boldsymbol{h} = \boldsymbol{w} \end{array}\right\}. \tag{19}$$

Here a hat is put over $\mathbf{z}_r$ to signify its estimate nature, since the observation vector $\mathbf{y}$ is random.

On the other hand, it has been shown by Golub & Saunders (1970) that $\boldsymbol{h}$ and $\boldsymbol{q}$ must satisfy the following condition:

$$\boldsymbol{q}^T\boldsymbol{h} = 0, \quad \boldsymbol{q}, \ \boldsymbol{h} \geq 0. \tag{20}$$

From the first equality of (19), we have

$$\begin{aligned} \hat{\mathbf{z}}_r &= (\mathbf{B}^T\mathbf{P}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{P}\mathbf{y} + (\mathbf{B}^T\mathbf{P}\mathbf{B})^{-1}\mathbf{D}^T\boldsymbol{p} \\ &\quad +(\mathbf{B}^T\mathbf{P}\mathbf{B})^{-1}\mathbf{G}^T\boldsymbol{q} \\ &= \mathbf{N}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{y} + \mathbf{N}^{-1}\mathbf{D}^T\boldsymbol{p} + \mathbf{N}^{-1}\mathbf{G}^T\boldsymbol{q} \\ &= \hat{\mathbf{z}}_r^0 + \mathbf{N}^{-1}\mathbf{D}^T\boldsymbol{p} + \mathbf{N}^{-1}\mathbf{G}^T\boldsymbol{q}, \end{aligned} \tag{21}$$

where

$$\mathbf{N} = \mathbf{B}^T\mathbf{P}\mathbf{B}$$

and

$$\hat{\mathbf{z}}_r^0 = \mathbf{N}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{y},$$

which is actually the floating solution of the original LS problem (14).

Inserting (21) into the last two equations of (19) yields

$$\left.\begin{array}{rcl} \mathbf{D}\hat{\mathbf{z}}_r^0 + \mathbf{D}\mathbf{N}^{-1}\mathbf{D}^T\boldsymbol{p} + \mathbf{D}\mathbf{N}^{-1}\mathbf{G}^T\boldsymbol{q} &=& \boldsymbol{e} \\ \mathbf{G}\hat{\mathbf{z}}_r^0 + \mathbf{G}\mathbf{N}^{-1}\mathbf{D}^T\boldsymbol{p} + \mathbf{G}\mathbf{N}^{-1}\mathbf{G}^T\boldsymbol{q} &=& \boldsymbol{h} + \boldsymbol{w} \end{array}\right\}. \tag{22}$$

Hence $\boldsymbol{p}$ can be further expressed as

$$\boldsymbol{p} = \mathbf{M}^{-1}(\boldsymbol{e} - \mathbf{D}\hat{\mathbf{z}}_r^0) - \mathbf{M}^{-1}\mathbf{D}\mathbf{N}^{-1}\mathbf{G}^T\boldsymbol{q}, \quad (23)$$

where $\mathbf{M} = \mathbf{D}\mathbf{N}^{-1}\mathbf{D}^T$.
Substituting $\boldsymbol{p}$ of (23) in the second equation of (22), we have

$$\mathbf{G}_1\boldsymbol{q} + \boldsymbol{w}_1 = \boldsymbol{h}, \quad (24)$$

where

$$\mathbf{G}_1 = \mathbf{G}\mathbf{N}^{-1}\mathbf{G}^T - \mathbf{G}\mathbf{N}^{-1}\mathbf{D}^T\mathbf{M}^{-1}\mathbf{D}\mathbf{N}^{-1}\mathbf{D}^T,$$

$$\boldsymbol{w}_1 = \mathbf{G}\hat{\mathbf{z}}_r^0 + \mathbf{G}\mathbf{N}^{-1}\mathbf{D}^T\mathbf{M}^{-1}(\boldsymbol{e} - \mathbf{D}\hat{\mathbf{z}}_r^0) - \boldsymbol{w}.$$

Equation (24) is a linear complementarity problem, together with the zero constraint of (20). Solution methods for (24) can be found in van de Panne (1975) or Kojima et al. (1991). It is thus omitted here.

4 Validating the solved GPS carrier phase integer ambiguities

In previous sections, we have discussed the solution methods to a variety of integer and mixed integer observation models. The purpose is to obtain the global optimal solution of the integer ambiguities. If the observations contained no (random or systematic) errors (in an idealized environment), the global solution would be thus the "correct" or true one and no further validation would be required. In reality, however, GPS observables are affected by a number of (random and systematic) errors (see e.g. Lachapelle 1990). These errors will all contribute to the estimated global solution of the integer variables, and make it biased and uncertain. On the other hand, precise positioning and navigation at the accuracy of centimetre level allows almost no bias nor uncertainty in integer ambiguities. Therefore, much effort has been made to reduce to a maximum degree the biases and uncertainty of the ambiguities.
Random errors in GPS observables will be propagated into the uncertainty of the estimated GPS integer ambiguities. Increasing the number of observations by tracking to more satellites or collecting data for a longer period of time will reduce the uncertainty significantly. In other words, the dispersion of the estimated GPS integer ambiguities will become smaller and smaller. If there were no systematic errors, the global solution to (13) from the observations up to the present epoch should be correct and no validation would be needed. GPS surveying has been influenced by systematic errors in reality, however. For instance, significant error sources are probably due to signal multipath and residual ionosphere and troposphere effects, which behave systematically and can last for a period of time. Various GPS systematic error sources will result in the biases of the globally estimated GPS ambiguities from the correct (true) values, which can last for a certain period of time and has been often experienced. In order to avoid wrong ambiguities due to systematic and random errors, the resolved ambiguities have to be monitored and validated.
A number of criteria have been proposed to validate the solved GPS carrier phase integer ambiguities. They are mainly based on two types of information: the residuals of the carrier phase observables and position solutions from the pseudoranges (see e.g. Frei & Butler 1990; Chen & Lachapelle 1994; Abidin 1993; Landau & Euler 1992). One criterion that is most widely employed to validate the estimated ambiguities is the so-called ratio statistic, which is the ratio of the second minimum sum of squared residuals to the globally minimum sum of squared residuals, i.e.

$$F = \hat{\sigma}_{2nd}^2 / \hat{\sigma}_{min}^2 \leq c.$$

This test quantity is simple and intuitive. Very often, it is treated as if it were an F-statistic, and applied with success in practice. The constant c involved is selected more empirically than theoretically, ranging from 1.5 to 5. It is noted that the F quantity may be vulnerable to systematic errors or blunders in observations.
The second residual-based statistic is the sum of squared residuals, i.e.

$$\mathbf{V}^T\mathbf{P}\mathbf{V}/\sigma_0^2 \leq \chi^2(1-\alpha),$$

where $\mathbf{V}$ is the residual vector of the carrier phase observables, $\mathbf{P}$ is the weight matrix, σ_0^2 is the empirical known variance of the carrier phase observables, $\chi^2(1-\alpha)$ is the percentile value of the χ^2 distribution at the significance

Figure 1: The distance (km) from the ship to the reference station

level α. We know that $\mathbf{V}^T\mathbf{PV}$ can serve as an accurate estimator of the variance component of the carrier phase observables, only if there exist a great number of redundant observations and if the observations are contaminated only by random errors. Thus one should use $\mathbf{V}^T\mathbf{PV}$ to validate or reject any set of ambiguities, with in mind that it may be significantly affected by systematic multipath and residual atmosphere errors.

The third criterion widely used is to compare the position solution from pseudorange observables with that derived with potential sets of ambiguities. If the accuracy of the pseudoranges is sufficiently good, it can be very helpful in constraining the size of the searching window of the ambiguities and thus help sort out the correct ambiguity solution. Since the accuracy of pseudoranges is much poorer than that of carrier phase observables, the pseudorange-derived position is mainly used to reject some non-promising sets of ambiguities. Other criteria can be found in Abidin (1993).

Almost all of the current criteria are designed to validate or test globally the whole set of ambiguities. As soon as a single ambiguity is concerned, some techniques need to be developed. A simple and intuitive criterion is the repeatability of the ambiguities. Suppose that given the pseudoranges and carrier phase observables at each epoch, we can obtain the optimal (integer) solution of the ambiguities for every corresponding epoch. It is reasonable to require that the correct set of ambiguities be able to repeat themselves for a period of time. By checking the repeatability of each ambiguity epoch by epoch, we are emphasizing the local test of each integer component of an ambiguity vector (set), which is actually the first validation criterion for our experiments in section 5. The second criterion used is that the maximum carrier phase residual should be smaller than a threshold. We set 4 cm to check the maximum L1 double differenced carrier phase residual.

5 Experiments

The data were collected in November 1994 off the coast of Vancouver. The ship used was a research vessel of the Canadian Department of National Defence. In this experiment, we will use the dual frequency carrier phase and C/A code pseudorange measurements from Ashtech Z-12 receivers. The experiment lasted for more than 4 hours. At the beginning of the experiment, the ship was separated from the reference station by about 600 metres, and about 27 minutes of stationary data were collected with a data sampling rate of 1 second. It then started to move, and was about 80 kilometres from the reference station at the end of the experiment, as shown in Figure 1. Plotted in Figure 2 was

Figure 2: The number of common satellites in view for computation

Figure 3: The number of epochs of observations for ambiguity resolution

the number of satellites above a marking angle of 10 degrees.

With this data set at hand, we have completed the following experiments with the widelane observations: (1) reliability test of the algorithm; For this purpose, we started the ambiguity resolution at the GPS time of 163700 seconds and ran through the whole data set of more than 4 hours. After that, we shifted the starting time from the previous one by 100 seconds and ran the ambiguity resolution again. The same procedure was repeated. In total, there are 154 such experiments. Generally, ambiguity resolution took a fraction of a second to complete on our Pentium 90 PC. Unprecedented advance in computer technology will further merit the GPS community considerably. (2) sensitivity analysis of the ambiguity resolution to carrier phase noise; In this experiment, we set the noise of code observables to 1.4 m and the noise of carrier phase observables to 2, 3, 4, 7 and 10 cm, respectively, which are used here simply to show how the standard deviation of double differenced carrier phase observations influence GPS ambiguity resolution. The procedure described in the reliability test was then repeated with each noise level.

Before talking about success rate, we have to justify the correct set of resolved ambiguities. In this experiment, we found and justified the correct ambiguity set, because the residuals of all the L1 double differenced carrier phase observables are small over 4 hours, as seen in Figs.5 and 6. If one of the ambiguities had been wrongly resolved, we should have witnessed messy plots of the L1 residuals, because a mistake in GPS ambiguities is actually equivalent to a gross error in the corresponding observation. The success rates of the methodology, defined as the ratio of the number of success in resolving the ambiguity unknowns correctly to the total experiment number of resolving the ambiguity unknowns, is given for the data set in Table 1. Given 2, 3 and 4 centimetres for the accuracy of double difference phase observables, the success rates of correctly fixing the ambiguities reach 100% if the separation of the remote station to the reference station is less than 65 km, but are still above 90% for the total of 154 trials. Further computation should show that the use of the method will be risky at these levels of accuracy, if the separation is more than 65 km. If the accuracy is given of 7 cm, the method performs excellently. The success rate reaches 98.7% for all the runs, and the average number of epochs for fixing the ambiguities correctly is 249. The best performance is 17 seconds (or epochs) for a baseline of over 60 km. The case of 10 cm is similar to the first three levels of accuracy, however.

The numbers of epochs of observations for correctly fixing the ambiguities are plotted in Figures 3 and 4. The numbers of epochs of observations needed vary significantly. However, it seems that the patterns for these four levels of accuracy are very similar, which might be related to the quality of the data set. Figures 5 and 6 show the residuals of six L1 double difference phase observables, which are computed after converting the widelane ambiguities into the L1 frequency ambiguities. Generally speaking, the observations are of good quality for the first hour of journey and then become noisy and probably contaminated by some periodical errors. In particular, the number of satellites in view remain five for a great part of the second half voyage, which, together with the noisy observations, may well explain that a lot more epochs of observations are needed to correctly resolve the ambiguities.

6 Conclusions

This study has further investigated integer programming for use in GPS ambiguity resolution with real marine data and thus compliments our earlier theoretical paper on this topic (Xu, Cannon & Lachapelle 1995). A major advantage of using integer programming techniques is that a global solution is always guaranteed and no bounds are necessary for the ambiguities to be searched. Since systematic and random errors may significantly deviate the global solution from the correct one, the resolved ambiguities have to be subject to monitoring and validation. It should be noted, however, that good prior information on the bounds for GPS carrier phase ambiguities will further speed up the searching procedure by integer programming. In the case of quadratic integer programming, the solution will solely depend on the centre and the shape of the ellipsoid, but not the size

Figure 4: The number of epochs of observations for ambiguity resolution

Figure 5: The residuals of L1 double difference phase observables

Table 1: The success rates with the ship data set

$\sigma_{\nabla\Delta\Phi}$	Numbers of trials	Rate	Numbers of trials (below 65 km)	Rate (below 65 km)
2cm	154	90.260	136	100.000
3cm	154	90.260	136	100.000
4cm	154	90.260	136	100.000
7cm	154	98.701	135	100.000
10cm	154	88.961	135	98.519

Figure 6: The residuals of L1 double difference phase observables

of it. With the help of validation criteria, we have tested an integer programming algorithm with a shipborne data set. The success rates for correctly resolving the ambiguities are quite high. However, the number of epochs of observations needed seems on the higher side, which might be related to the noisy observations and a relatively small number of satellites in view on one hand, and to the validation criterion on the other hand. The experiment also shows that the accuracy of the carrier phase observables can significantly influence the ambiguity resolution.

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