Universität Stuttgart

Schriftenreihe der Institute des Studiengangs Geodäsie und Geoinformatik

Technical Reports
Department of Geodesy and Geoinformatics

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Continuum Mechanics for Geophysicists and Geodesists
Part I: Basic Theory

ISSN 0933-2839

Report Nr. 1999.2

CONTINUUM MECHANICS

for

GEOPHYSICISTS AND GEODESISTS

Part I. BASIC THEORY

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Lecture Notes

1999

Preface

This text is suitable for a one-semester course on continuum mechanics. It is based on notes from undergraduate courses that I have taught over the last few years. The material is intended for use by undergraduate students of physics with a year or more of college calculus behind them.

I would like to thank Erik Grafarend, Ctirad Matyska, Detlef Wolf and Jiří Zahradník, whose interest encouraged me to write this text. I would also like to thank my oldest son Zdeněk who plotted all figures embedded in the text.

Readers of this text are encouraged to contact me with their comments, suggestions, and questions. I would be very happy to hear what you think I did well and I could do better. My e-mail address is zdenek@hervam.troja.mff.cuni.cz and a full mailing address is found on the title page.

Zdeněk Martinec

Contents

1. STRAIN

1.1 Particles, configurations, deformation, and motion

In continuum mechanics we consider *material bodies* in the form of solids, liquids, and gases. Let us begin by describing the model we use to represent such bodies. For this purpose we define the material body as the set of elements, called *particles* or *material points*, which can be put into one-to-one correspondence with the points of a regular region of physical space. Note that whereas a "particle" of classical mechanics has an assigned mass, a "continuum particle" is essentially a material point for which a density is defined.

The specification of the position of all particles of a material body with respect to a fixed origin at some instant of time is said to define the *configuration* of the body at that instant. We give special meaning to certain configurations of the body.

In particular, we single out a *reference configuration* from which all displacements are reckoned. For the purpose it serves, the reference configuration needs not be one which the body ever actually occupies. We choose, however, the *initial configuration*, that is, the one which the body occupies at time $t = 0$, as the reference configuration and the ensuing deformations and motions are related to it. The material points of a continuous medium at the reference configuration occupy a region B which consists of the material volume V and its surface S. The position of a material point P in region B may be denoted by a rectangular coordinate system X_K, $K = 1, 2, 3$, or by a vector $\boldsymbol{X}$ that extends from an origin O of the coordinates to the point P (Figure 1.1).

After deformation takes place, at the current time t, the material points of $B + V$ occupy the *current configuration* in a region b consisting of a spatial volume v and its surface s. In this deformed state, a material point may occupy a spatial point p. We may locate p by a vector $\boldsymbol{x}$ extending from the origin o of a new coordinate frame or by a set of rectangular coordinates x_k, $k = 1, 2, 3$. Following the current terminology, we shall call X_K the *material* or *Lagrangian* coordinates and x_k the *spatial* or *Eulerian* coordinates. In next considerations we assume that these two coordinate frames, one for the undeformed body and one for the deformed body, are nonidentical. Let us emphasize that the material coordinates are used in conjunction with the reference configuration only and that the spatial coordinates serve for all other configurations. As has been remarked already, the material coordinates are therefore time independent.

Under the influence of the external loads, the body B moves and deforms. The deformation and motion of the body carries various material points through various spatial positions. This is expressed by a one-parameter family of mappings

$$x_k = \chi_k(X_1, X_2, X_3, t) \qquad \text{or} \qquad x_k = \chi_k(X_K, t) \qquad k = 1, 2, 3 \ , \tag{1.1}$$

or, conversely,

$$X_K = \chi_K^{-1}(x_1, x_2, x_3, t) \qquad \text{or} \qquad X_K = \chi_K^{-1}(x_k, t) \qquad K = 1, 2, 3 \ . \tag{1.2}$$

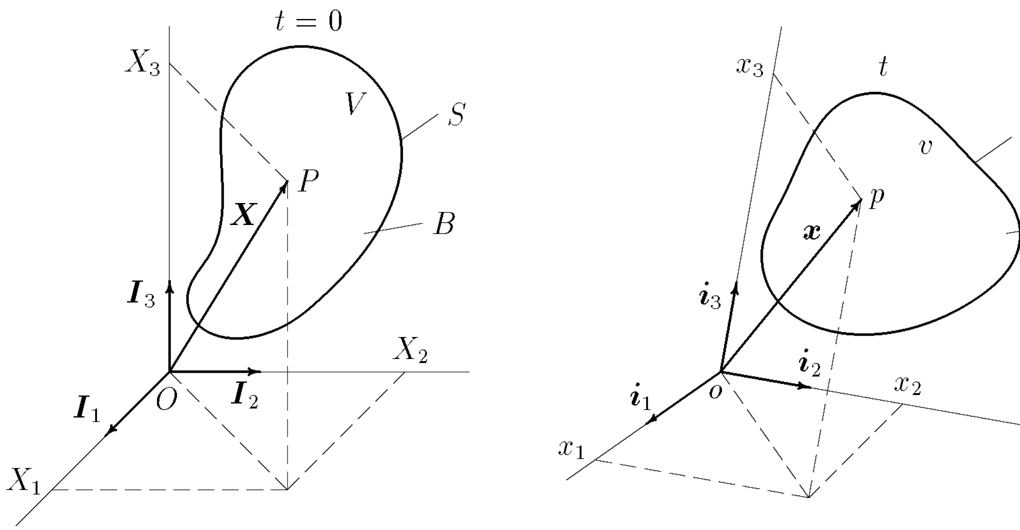

Figure 1.1. Coordinate system for an undeformed body B and a deformed body b.

For brevity, we may also write these in coordinate-free (or symbolic) notation

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t), \qquad \boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t) \; . \tag{1.3}$$

Equation (1.1) states that the motion takes a material point P in the reference configuration B to a current position p in the current configuration b at time t. The inverse motion (1.2) states the inverse phenomenon, namely, that we can trace the material point occupying the current position $\boldsymbol{x}$ at time t to its original position $\boldsymbol{X}$. It is common practice in continuum mechanics to write these equations in the alternative forms

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t), \qquad \boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t) \tag{1.4}$$

with understanding that the symbol $\boldsymbol{x}$ (or, $\boldsymbol{X}$) on the right-hand sides of these equations represents the *function* whose arguments are $\boldsymbol{X}$ (or, $\boldsymbol{x}$) and t while the same symbol on the left-hand sides represents the *value* of the function, that is a point in the space. We shall use this notation frequently in the text that follows.

We assume that the mappings (1.3) are single-valued and possess continuous partial derivatives with respect to their arguments for whatever order is desired, except possibly at some singular points, curves, and surfaces. Moreover, each member of (1.3) is the unique inverse of the other in a neighborhood of the material point P. This assumption is known as the *axiom of continuity*. It expresses the fact that the matter is *indestructible*, that is, no region of positive, finite volume of matter is deformed into a zero or infinite volume. Another implication of this axiom is that the matter is *impenetrable*, that is, the motion caries every region into a region, every surface into a surface, and every curve into a curve. One portion of matter never penetrates into another. In practice, there are cases in which this axiom is violated. For example, the material may break or may transmit shock and other types of discontinuities. Special attention must be given to these. The axiom of continuity is secured through the well-known implicit function theorem.

THEOREM (IMPLICIT FUNCTION). If, for a fixed time t the functions $x_k(X_K, t)$ are continuous and possess continuous first-order partial derivatives with respect to X_K in a neighborhood $|X'_K - X_K| < \Delta$ of the point P, and if the jacobian

$$j := \det\left(\frac{\partial x_k}{\partial X_K}\right) \tag{1.5}$$

does not vanish there, then a unique inverse of the form (1.3) exists in a neighborhood of $|x'_k - x_k| < \delta$ of a point p at time t.

If we determine $\boldsymbol{x}$ as a function of time for each material point P (that is, given by $\boldsymbol{X}$), we can construct the new shape and position of the body at each time t relative to that at $t = 0$. This enables us to calculate the change of length between any two points and the change of angle between any two directions. The ultimate goal is to relate these deformations to the external effects (for example, external forces, thermal changes). With the possession of such a knowledge, one may hope to design machines and buildings or analyse the existing natural or man-made materials and structures so that not only can failure be avoided, but also maximum performance can be achieved. *Thus, the subject of continuous media deals, in essence, with the determination of the explicit form of (1.3) when the external effects and initial and boundary conditions of the body are known.*

The quantities associated with the undeformed body B will be denoted by capital letters, and those associated with the deformed body b by lower case letters. When these quantities are referred to coordinates X_K, their indices will be majuscules; and when they are referred to x_k, their indices will be minuscules. For example, a vector $\boldsymbol{V}$ in B referred to X_K will have components V_K, and referred to x_k will have the components V_k. Conversely, a vector $\boldsymbol{v}$ in b referred to X_K and x_k will have components denoted by v_K and v_k, respectively.

1.2 Base vectors, shifters

The position vectors $\boldsymbol{X}$ of a point P in B and $\boldsymbol{x}$ of p in b, respectively, referred to rectangular coordinates X_K and x_k are given by

$$\boldsymbol{X} = X_K \boldsymbol{I}_K, \qquad \boldsymbol{x} = x_k \boldsymbol{i}_k \ , \tag{1.6}$$

where $\boldsymbol{I}_K$ and $\boldsymbol{i}_k$ are, respectively the *unit base vectors* in Figure 1.1. Henceforth, we employ the usual summation convention over repeated indices, that is,

$$\boldsymbol{X} = X_K \boldsymbol{I}_K = X_1 \boldsymbol{I}_1 + X_2 \boldsymbol{I}_2 + X_3 \boldsymbol{I}_3 \ . \tag{1.7}$$

Infinitesimal differential vectors $d\boldsymbol{X}$ at P and $d\boldsymbol{x}$ at p may be expressed as

$$d\boldsymbol{X} = dX_K \boldsymbol{I}_K, \qquad d\boldsymbol{x} = dx_k \boldsymbol{i}_k \ . \tag{1.8}$$

Thus the squares of the elements of length in B and b are, respectively,

$$dS^2 = d\boldsymbol{X} \cdot d\boldsymbol{X} = \delta_{KL} dX_K dX_L = dX_K \, dX_K \ , \tag{1.9}$$

$$ds^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} = \delta_{kl} dx_k dx_l = dx_k \, dx_k \ ,$$

where

$$\delta_{KL} = \boldsymbol{I}_K \cdot \boldsymbol{I}_L, \qquad \delta_{kl} = \boldsymbol{i}_k \cdot \boldsymbol{i}_l \tag{1.10}$$

are the *Kronecker symbols*, which are equal to 1 when the two indices are equal and zero otherwise.

When the two rectangular coordinates are not identical, we shall also express a vector in one frame (say x_k) in terms of its projection in another frame (say X_K). For any vector we may write

$$\boldsymbol{v} = \boldsymbol{V} = v_k \boldsymbol{i}_k = V_K \boldsymbol{I}_K \ . \tag{1.11}$$

By taking the scalar products of this by $\boldsymbol{I}_L$ and $\boldsymbol{i}_l$ we find the components V_K of $\boldsymbol{v}$ in X_K and the components v_k of $\boldsymbol{v}$ in x_k:

$$V_K = \boldsymbol{V} \cdot \boldsymbol{I}_K = \boldsymbol{v} \cdot \boldsymbol{I}_K = v_k \boldsymbol{i}_k \cdot \boldsymbol{I}_K = v_k \delta_{kK} \ , \tag{1.12}$$

where

$$\delta_{kK} = \delta_{Kk} =: \boldsymbol{i}_k \cdot \boldsymbol{I}_K \tag{1.13}$$

are called *shifters*. They are *not* a Kronecker symbol except when the two frames are identical. It is clear that (1.13) is none other than the cosine directors of the two frames of reference x_k and X_K. Note that the dual of (1.12) is

$$v_k = \delta_{kK} V_K \ , \qquad V_K = \delta_{Kk} v_k \ . \tag{1.14}$$

By carrying one of (1.12), (1.14) into the other, we find that

$$\delta_{Kk} \delta_{kL} = \delta_{KL}, \qquad \delta_{kK} \delta_{Kl} = \delta_{kl} \ . \tag{1.15}$$

Finally, if we substitute (1.14) into (1.11) and assume that the result must be valid for all vectors, we get

$$\boldsymbol{i}_k = \delta_{kK} \boldsymbol{I}_K \ , \qquad \boldsymbol{I}_K = \delta_{Kk} \boldsymbol{i}_k \ . \tag{1.16}$$

1.3 Deformation gradients and deformation tensors

From (1.1) and (1.2), for fixed time, we have

$$dx_k = x_{k,K} dX_K \ , \qquad dX_K = X_{K,k} dx_k \ , \tag{1.17}$$

where indices following a comma represent partial differentiation with respect to X_K, when they are majuscules, and with respect to x_k when they are minuscules, that is,

$$x_{k,K} := \frac{\partial x_k}{\partial X_K} \ , \qquad X_{K,k} := \frac{\partial X_K}{\partial x_k} \ . \tag{1.18}$$

The two sets of quantities defined by (1.18) are components of the *deformation gradient tensors*, or simply the *deformation gradients* $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$,

$$\boldsymbol{F}(\boldsymbol{X}, t) := x_{k,K}(\boldsymbol{X}, t) \, \boldsymbol{i}_k \boldsymbol{I}_K \ , \qquad \boldsymbol{F}^{-1}(\boldsymbol{x}, t) := X_{K,k}(\boldsymbol{x}, t) \, \boldsymbol{I}_K \boldsymbol{i}_k \ , \tag{1.19}$$

where $\boldsymbol{i}_k \boldsymbol{I}_K$ and $\boldsymbol{I}_K \boldsymbol{i}_k$ are the dyadic products of basis vectors $\boldsymbol{i}_k$ and $\boldsymbol{I}_K$. The deformation gradients $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$ are *two-point tensor fields*, i.e., their components transform like those of a vector under rotations of only one of two reference axes and like a two-point tensor when the two sets of axes are rotated independently. In symbolic notation, eqn.(1.17) appears in the form

$$d\boldsymbol{x} = \boldsymbol{F} \cdot d\boldsymbol{X} \ , \qquad d\boldsymbol{X} = \boldsymbol{F}^{-1} \cdot d\boldsymbol{x} \ , \tag{1.20}$$

where dot '$\cdot$' stands for the scalar product of vectors and tensors. The deformation gradient $\boldsymbol{F}$ can be thought of as a mapping of the infinitesimal vector $d\boldsymbol{X}$ of the reference configuration into the infinitesimal vector $d\boldsymbol{x}$ of the current configuration; the inverse mapping is performed by the spatial deformation gradient $\boldsymbol{F}^{-1}$.

Through the chain rule of partial differentiation it is clear that

$$x_{k,K} X_{K,l} = \delta_{kl} \ , \qquad X_{K,k} x_{k,L} = \delta_{KL} \ , \tag{1.21}$$

or in symbolic notation,

$$\boldsymbol{F} \cdot \boldsymbol{F}^{-1} = \boldsymbol{F}^{-1} \cdot \boldsymbol{F} = \boldsymbol{I} \ ,$$

where $\boldsymbol{I}$ is the identity tensor. Hence, the spatial deformation gradient $\boldsymbol{F}^{-1}$ is the inverse tensor of the deformation gradient $\boldsymbol{F}$. Each of the two sets of equations (1.21) consists of nine linear equations for the nine unknown $x_{k,K}$ or $X_{K,k}$. Since the jacobian is assumed not to vanish, a unique solution exists and, according to Cramer's rule of determinants, the solution for $X_{K,k}$ may be obtained in terms of $x_{k,K}$ as

$$X_{K,k} = \frac{\text{cofactor}\,(x_{k,K})}{j} = \frac{1}{2j} \epsilon_{KLM} \epsilon_{klm} x_{l,L} x_{m,M} \ , \tag{1.22}$$

where ϵ_{KLM} and ϵ_{klm} are the permutation symbols, and

$$j := \det\,(x_{k,K}) = \det \boldsymbol{F} = \frac{1}{3!} \epsilon_{KLM} \epsilon_{klm} x_{k,K} x_{l,L} x_{m,M} \ . \tag{1.23}$$

By differentiating (1.22) and (1.23) we get the following Jacobi identities:

$$\left(j X_{K,k}\right)_{,K} = 0 \ , \qquad \left(j^{-1} x_{k,K}\right)_{,k} = 0 \ , \tag{1.24}$$

$$\frac{\partial j}{\partial x_{k,K}} = \text{cofactor}\,(x_{k,K}) = j X_{K,k} \ . \tag{1.25}$$

If we substitute (1.17) into (1.8), we obtain

$$d\boldsymbol{X} = \boldsymbol{c}_k dx_k \ , \qquad d\boldsymbol{x} = \boldsymbol{C}_K dX_K \ , \tag{1.26}$$

where

$$\boldsymbol{c}_k(\boldsymbol{x}, t) := X_{K,k} \boldsymbol{I}_K \ , \qquad \boldsymbol{C}_K(\boldsymbol{X}, t) := x_{k,K} \boldsymbol{i}_k \ . \tag{1.27}$$

From these, we may solve $\boldsymbol{I}_K$ and $\boldsymbol{i}_k$. For example, multiply $(1.27)_1$ by $x_{k,l}$ and use $(1.21)_2$ to solve for $\boldsymbol{I}_K$, and similarly multiply $(1.27)_2$ by $X_{K,l}$ and use $(1.21)_1$ to solve $\boldsymbol{i}_k$. Thus

$$\boldsymbol{I}_K = x_{k,K} \boldsymbol{c}_k \ , \qquad \boldsymbol{i}_k = X_{K,k} \boldsymbol{C}_K \ . \tag{1.28}$$

In the same way as in (1.9), by use of (1.26), we see that

$$dS^2 = c_{kl} dx_k dx_l \ , \qquad ds^2 = C_{KL} dX_K dX_L \ , \tag{1.29}$$

where

$$c_{kl}(\boldsymbol{x}, t) := \boldsymbol{c}_k \cdot \boldsymbol{c}_l = \delta_{KL} X_{K,k} X_{L,l} = X_{K,k} X_{K,l} \ , \tag{1.30}$$
$$C_{KL}(\boldsymbol{X}, t) := \boldsymbol{C}_K \cdot \boldsymbol{C}_L = \delta_{kl} x_{k,K} x_{l,L} = x_{k,K} x_{k,L} \ ,$$

are, respectively, *Cauchy's deformation tensor* and *Green's deformation tensor*. They can be expressed in terms of deformation gradients,

$$\boldsymbol{c} = (\boldsymbol{F}^{-1})^T \cdot (\boldsymbol{F}^{-1}) \ , \qquad \boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} \ . \tag{1.31}$$

Both of these tensors are symmetric, that is $\boldsymbol{c} = \boldsymbol{c}^T$, $\boldsymbol{C} = \boldsymbol{C}^T$.

Two other equally important tensors are the *reciprocal tensors* $\boldsymbol{b}$ and $\boldsymbol{B}$ (known as the *Finger* and *Piola* deformation tensors, respectively) defined by

$$\boldsymbol{b}(\boldsymbol{x}, t) := \boldsymbol{F} \cdot \boldsymbol{F}^T \ , \qquad \boldsymbol{B}(\boldsymbol{X}, t) := (\boldsymbol{F}^{-1}) \cdot (\boldsymbol{F}^{-1})^T \ . \tag{1.32}$$

which, in indicial notation, read

$$b_{kl}(\boldsymbol{x}, t) := x_{k,K} x_{l,K} \qquad B_{KL}(\boldsymbol{X}, t) := X_{K,k} X_{L,k} \ .$$

They satisfy

$$\boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{b} = \boldsymbol{I} \ , \qquad \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{C} \cdot \boldsymbol{B} = \boldsymbol{I} \ , \tag{1.33}$$

which can be shown by mere substitution of (1.31) and (1.32).

We have been using the word *tensor* for quantities such as C_{KL} and c_{kl}. This term referees to a set of quantities that transform according to a certain definite law upon coordinate transformation. Suppose that coordinates X_K are transformed into X'_K according to

$$X_K = X_K(X'_1, X'_2, X'_3) \ . \tag{1.34}$$

The left-hand side of $(1.29)_2$ is independent of the coordinate transformations. If on the right-hand side we put

$$dX_K = \frac{\partial X_K}{\partial X'_M} dX'_M \ , \tag{1.35}$$

we get

$$ds^2 = C_{KL} \frac{\partial X_K}{\partial X'_M} \frac{\partial X_L}{\partial X'_N} dX'_M dX'_N = C'_{MN} dX'_M dX'_N \ . \tag{1.36}$$

Hence

$$C'_{MN}(\boldsymbol{X}', t) = C_{KL}(\boldsymbol{X}, t) \frac{\partial X_K}{\partial X'_M} \frac{\partial X_L}{\partial X'_N} \tag{1.37}$$

since dX'_M is arbitrary and $C_{KL} = C_{LK}$. Thus, knowing C_{KL} in one set of coordinates X_K, we can find the corresponding quantities in another set X'_K once the relations (1.34) between X_K and X'_K are given. Quantities that transform according to the law of transformation (1.37) are known as *absolute tensors*.

1.4 Rotation and stretch tensors

The basic properties of the local behavior of deformation emerge from the possibility to decompose the deformation into a rotation and stretch which, roughly speaking, is a change of the shape of the volume element. This decomposition is called the *polar decomposition of the deformation gradient*, and it is summarized in the following theorem.

A non-singular tensor $\boldsymbol{F}$ ($\det \boldsymbol{F} \neq 0$) admits the polar decompositions such that

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{R} \ , \tag{1.38}$$

where the particular factors have the following properties:

1. The tensor $\boldsymbol{R}$ is orthogonal, $\boldsymbol{R} \cdot \boldsymbol{R}^T = \boldsymbol{R}^T \cdot \boldsymbol{R} = \boldsymbol{I}$, i.e., $\boldsymbol{R}$ is a *rotation tensor*.
2. The tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric and positive definite.
3. $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{R}$ are uniquely determined.
4. The eigenvalues of $\boldsymbol{U}$ and $\boldsymbol{V}$ are identical; if $\boldsymbol{e}$ is an eigenvector of $\boldsymbol{U}$, then $\boldsymbol{R} \cdot \boldsymbol{e}$ is an eigenvector of $\boldsymbol{V}$.

As a preliminary to proving these statements, we note that an arbitrary tensor $\boldsymbol{T}$ is positive definite if $\boldsymbol{v} \cdot \boldsymbol{T} \cdot \boldsymbol{v} > 0$ for all vectors $\boldsymbol{v} \neq \boldsymbol{0}$. A necessary and sufficient condition for $\boldsymbol{T}$ to be positive definite is that all its eigenvalues be positive. In this regard, consider the Green deformation tensor $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$. Since $\boldsymbol{F}$ is assumed to be non-singular ($\det \boldsymbol{F} \neq 0$) and $\boldsymbol{F} \cdot \boldsymbol{v} \neq \boldsymbol{0}$ if $\boldsymbol{v} \neq \boldsymbol{0}$, it follows that $(\boldsymbol{F} \cdot \boldsymbol{v}) \cdot (\boldsymbol{F} \cdot \boldsymbol{v})$ is a sum of squares and hence greater than zero. Thus

$$0 < (\boldsymbol{F} \cdot \boldsymbol{v}) \cdot (\boldsymbol{F} \cdot \boldsymbol{v}) = \boldsymbol{v} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{C} \cdot \boldsymbol{v} \ ,$$

and $\boldsymbol{C}$ is positive definite. By the same arguments, we may show that Finger's deformation tensor $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ is also positive definite.

The positive roots of $\boldsymbol{C}$ and $\boldsymbol{b}$ define two tensors $\boldsymbol{U}$ and $\boldsymbol{V}$,

$$\boldsymbol{U} := \sqrt{\boldsymbol{C}} = \sqrt{\boldsymbol{F}^T \cdot \boldsymbol{F}} \ , \qquad \boldsymbol{V} := \sqrt{\boldsymbol{b}} = \sqrt{\boldsymbol{F} \cdot \boldsymbol{F}^T} \ . \tag{1.39}$$

The tensors $\boldsymbol{U}$ and $\boldsymbol{V}$, called the *right* and *left stretch tensors*, are symmetric, positive definite and are uniquely determined. Next, two tensors $\boldsymbol{R}$ and $\tilde{\boldsymbol{R}}$ are defined by

$$\boldsymbol{R} := \boldsymbol{F} \cdot \boldsymbol{U}^{-1} \ , \qquad \tilde{\boldsymbol{R}} := \boldsymbol{V}^{-1} \cdot \boldsymbol{F} \ . \tag{1.40}$$

We recognize that both are orthogonal since by definition we have

$$\boldsymbol{R} \cdot \boldsymbol{R}^T = \left(\boldsymbol{F} \cdot \boldsymbol{U}^{-1}\right) \cdot \left(\boldsymbol{F} \cdot \boldsymbol{U}^{-1}\right)^T = \boldsymbol{F} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \left(\boldsymbol{U}^2\right)^{-1} \cdot \boldsymbol{F}^T =$$

$$= \boldsymbol{F} \cdot \left(\boldsymbol{F}^T \cdot \boldsymbol{F}\right)^{-1} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \boldsymbol{F}^{-1} \cdot (\boldsymbol{F}^T)^{-1} \cdot \boldsymbol{F}^T = \boldsymbol{I} \ . \tag{1.41}$$

A similar proof holds for $\tilde{\boldsymbol{R}}$. So far we have demonstrated two decompositions $\boldsymbol{F} = \boldsymbol{R}\cdot\boldsymbol{U} = \boldsymbol{V}\cdot\tilde{\boldsymbol{R}}$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric, positive definite and $\boldsymbol{R}$ and $\tilde{\boldsymbol{R}}$ are orthogonal. From

$$\boldsymbol{F} = \boldsymbol{V}\cdot\tilde{\boldsymbol{R}} = \left(\tilde{\boldsymbol{R}}\cdot\tilde{\boldsymbol{R}}^T\right)\cdot\boldsymbol{V}\cdot\tilde{\boldsymbol{R}} = \tilde{\boldsymbol{R}}\cdot\left(\tilde{\boldsymbol{R}}^T\cdot\boldsymbol{V}\cdot\tilde{\boldsymbol{R}}\right) = \tilde{\boldsymbol{R}}\cdot\tilde{\boldsymbol{U}} \tag{1.42}$$

it may be concluded that there might be two decompositions of $\boldsymbol{F}$, namely $\boldsymbol{F} = \boldsymbol{R}\cdot\boldsymbol{U}$ and $\boldsymbol{F} = \tilde{\boldsymbol{R}}\cdot\tilde{\boldsymbol{U}}$. However, if this were true we were forced to conclude that $\boldsymbol{C} = \boldsymbol{F}^T\cdot\boldsymbol{F} = \tilde{\boldsymbol{U}}^2 = \boldsymbol{U}^2$, whence follows that $\tilde{\boldsymbol{U}} = \boldsymbol{U}$, because of the uniqueness of the positive root. This implies $\tilde{\boldsymbol{R}} = \boldsymbol{R}$; consequently, $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{R}$ are unique.

Finally, we assume $\boldsymbol{e}$ and λ to be an eigenvector and eigenvalue of $\boldsymbol{U}$. Then, we have $\lambda\boldsymbol{e} = \boldsymbol{U}\cdot\boldsymbol{e}$, as well as $\lambda\boldsymbol{R}\cdot\boldsymbol{e} = (\boldsymbol{R}\cdot\boldsymbol{U})\cdot\boldsymbol{e} = (\boldsymbol{V}\cdot\boldsymbol{R})\cdot\boldsymbol{e} = \boldsymbol{V}\cdot(\boldsymbol{R}\cdot\boldsymbol{e})$. Thus λ is also eigenvalue of $\boldsymbol{V}$ and $\boldsymbol{R}\cdot\boldsymbol{e}$ is an eigenvector. This completes the proof of the theorem.

Equation $d\boldsymbol{x} = \boldsymbol{F}\cdot d\boldsymbol{X}$ shows that the deformation gradient $\boldsymbol{F}$ can be thought of as a mapping of the infinitesimal vector $d\boldsymbol{X}$ of the reference configuration into the infinitesimal vector $d\boldsymbol{x}$ of the current configuration. The theorem of polar decomposition replaces the linear transformation $d\boldsymbol{x} = \boldsymbol{F}\cdot d\boldsymbol{X}$ by two sequential transformations, by rotation and stretching, where the sequence of these two steps may be interchanged, as illustrated in Figure 1.2. The combination of rotation and stretching corresponds to the multiplication of two tensors, namely, $\boldsymbol{R}$ and $\boldsymbol{U}$ or $\boldsymbol{V}$ and $\boldsymbol{R}$,

$$d\boldsymbol{x} = (\boldsymbol{R}\cdot\boldsymbol{U})\cdot d\boldsymbol{X} = (\boldsymbol{V}\cdot\boldsymbol{R})\cdot d\boldsymbol{X}\ . \tag{1.43}$$

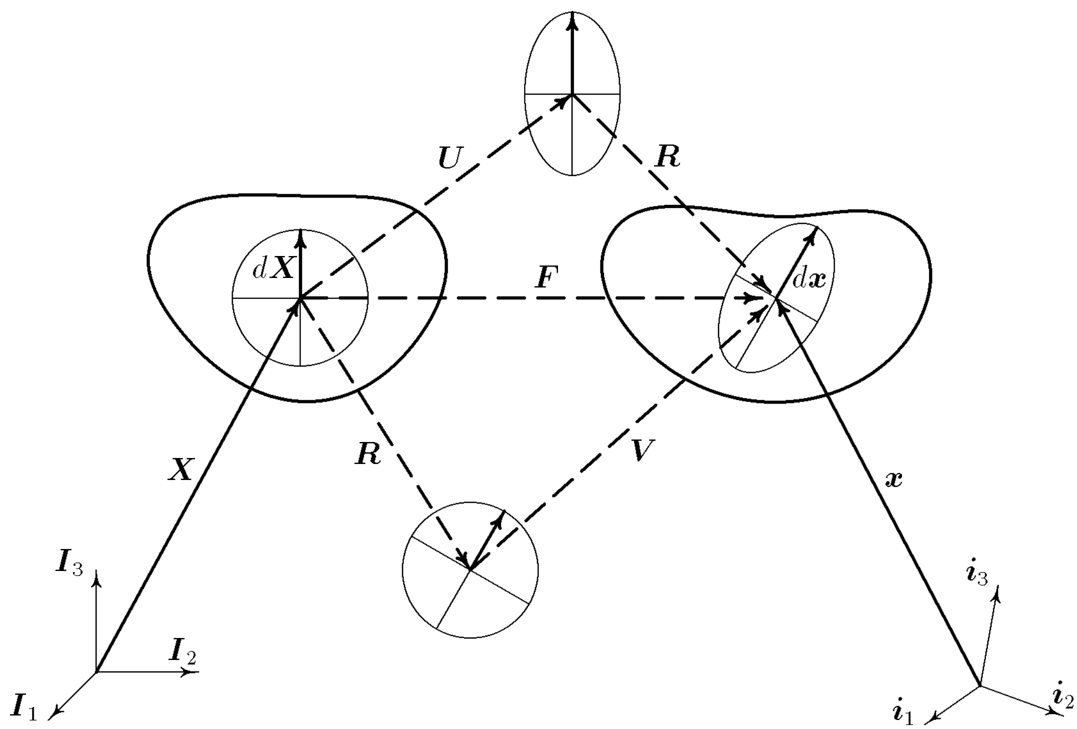

Figure 1.2. Polar decomposition.

However, $\boldsymbol{R}$ should not be understood as a rigid body rotation since, in general case, it varies from point to point. Thus the polar decomposition theorem reflects only a local property of motion.

1.5 Strain tensors and displacement vector

From (1.9) and (1.29) we have two different expressions of the squares of element of length, dS^2 in the undeformed body and ds^2 in the deformed body,

$$dS^2 = \delta_{KL} dX_K dX_L = c_{kl} dx_k dx_l \ , \tag{1.44}$$

$$ds^2 = C_{KL} dX_K dX_L = \delta_{kl} dx_k dx_l \ .$$

The difference $ds^2 - dS^2$ for the same material points in B and b is a measure of the change of length. When this difference vanishes for any two neighboring points, the deformation has not changed the distance between the pair. When it is zero for all points in the body, the body has undergone only a *rigid displacement*.

Thus, from (1.44) for this difference, in the same coordinate system, we obtain

$$ds^2 - dS^2 = 2E_{KL}(\boldsymbol{X}, t) dX_K dX_L = 2e_{kl}(\boldsymbol{x}, t) dx_k dx_l \ , \tag{1.45}$$

or, in symbolic notation,

$$ds^2 - dS^2 = d\boldsymbol{X} \cdot 2\boldsymbol{E} \cdot d\boldsymbol{X} = d\boldsymbol{x} \cdot 2\boldsymbol{e} \cdot d\boldsymbol{x} \ ,$$

where

$$E_{KL} := \frac{1}{2}\left(C_{KL} - \delta_{KL}\right) \ , \qquad e_{kl} := \frac{1}{2}\left(\delta_{kl} - c_{kl}\right) \ , \tag{1.46}$$

or

$$2\boldsymbol{E} := \boldsymbol{C} - \boldsymbol{I} \ , \qquad 2\boldsymbol{e} := \boldsymbol{I} - \boldsymbol{c} \ ,$$

are called *Lagrangian* and *Eulerian strain tensors*, respectively. Clearly, when either vanishes, $ds^2 = dS^2$.

From (1.45) we can see that

$$E_{KL} = e_{kl} x_{k,K} x_{l,L} \ , \qquad e_{kl} = E_{KL} X_{K,k} X_{L,l} \ , \tag{1.47}$$

which exhibit the fact that both E_{KL} and e_{kl} are second-order absolute tensors.

We may express the strain components in terms of the *displacement vector* $\boldsymbol{u}$ that extends from a material points P in the undeformed body to its spatial position in the deformed body, as illustrated in Figure 1.3:

$$\boldsymbol{u} := \boldsymbol{x} - \boldsymbol{X} + \boldsymbol{b} = x_l \boldsymbol{i}_l - X_L \boldsymbol{I}_L + \boldsymbol{b} \ . \tag{1.48}$$

The displacement vector may be expressed in terms of its Lagrangian and Eulerian components U_K and u_k as

$$\boldsymbol{u} = \boldsymbol{U} = u_l \boldsymbol{i}_l = U_L \boldsymbol{I}_L \ . \tag{1.49}$$

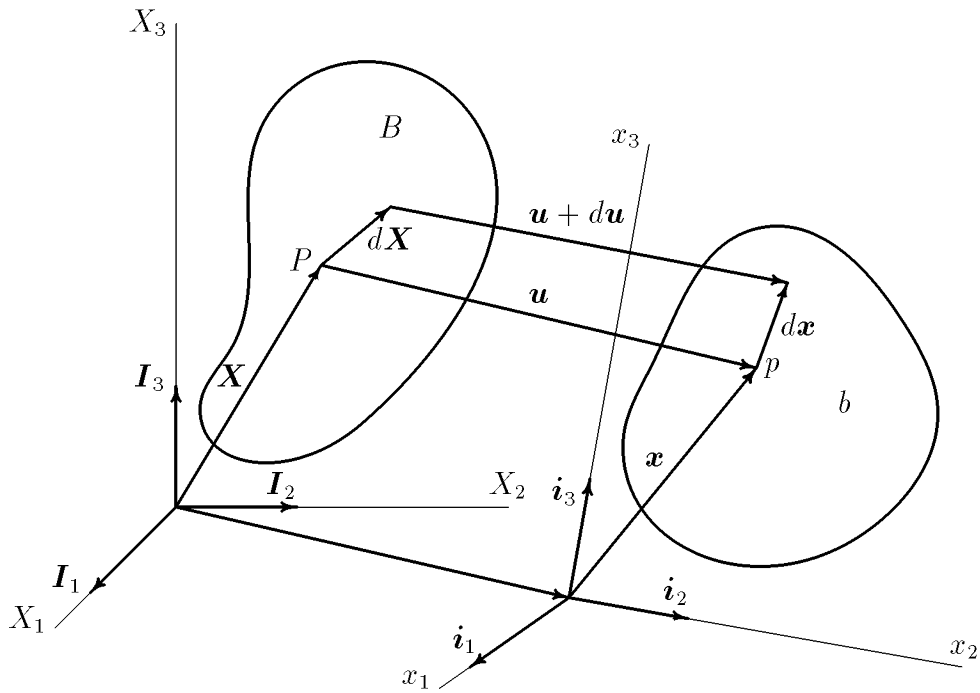

Figure 1.3. Displacement vector.

By taking the scalar product of both sides of (1.48) by $\boldsymbol{i}_k$ or $\boldsymbol{I}_K$, we obtain

$$u_k = x_k - \delta_{kL} X_L + b_k \;, \qquad U_K = \delta_{Kl} x_l - X_K + B_K \;, \tag{1.50}$$

where $b_k = \boldsymbol{b} \cdot \boldsymbol{i}_k$ and $B_K = \boldsymbol{b} \cdot \boldsymbol{I}_K$. Here again we can see the appearance of shifters. Differentiating the last equation with respect to X_M, we have

$$U_{K,M} = \delta_{Kl} x_{l,M} - \delta_{KM} \qquad \text{or} \qquad \boldsymbol{H} = \boldsymbol{F}^T - \boldsymbol{I} \;, \tag{1.51}$$

where $\boldsymbol{H}$ is the *displacement gradient tensor*, or simply *displacement gradient*,

$$\boldsymbol{H}(\boldsymbol{X}, t) := \operatorname{Grad} \boldsymbol{U}(\boldsymbol{X}, t) \;. \tag{1.52}$$

We now calculate the strain tensors. First from (1.27) and (1.48) we have

$$\boldsymbol{C}_K = \frac{\partial \boldsymbol{x}}{\partial X_K} = \frac{\partial \boldsymbol{X}}{\partial X_K} + \frac{\partial \boldsymbol{u}}{\partial X_K} = \boldsymbol{I}_K + U_{M,K} \boldsymbol{I}_M \;, \tag{1.53}$$

$$\boldsymbol{c}_k = \frac{\partial \boldsymbol{X}}{\partial x_k} = \frac{\partial \boldsymbol{x}}{\partial x_k} - \frac{\partial \boldsymbol{u}}{\partial x_k} = \boldsymbol{i}_k - u_{m,k} \boldsymbol{i}_m \;,$$

which yields

$$C_{KL} = \boldsymbol{C}_K \cdot \boldsymbol{C}_L = (\boldsymbol{I}_K + U_{M,K} \boldsymbol{I}_M) \cdot (\boldsymbol{I}_L + U_{N,L} \boldsymbol{I}_N) =$$

$$= \delta_{KL} + U_{K,L} + U_{L,K} + U_{M,K}U_{M,L} \ , \tag{1.54}$$

or, in symbolic notation,

$$\boldsymbol{C} = \boldsymbol{I} + \boldsymbol{H} + \boldsymbol{H}^T + \boldsymbol{H} \cdot \boldsymbol{H}^T \ .$$

Substituting this into $(1.46)_1$, we obtain

$$E_{KL} = \frac{1}{2}\left(U_{K,L} + U_{L,K} + U_{M,K}U_{M,L}\right) \qquad \text{or} \qquad \boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{H} + \boldsymbol{H}^T + \boldsymbol{H} \cdot \boldsymbol{H}^T\right) \ . \tag{1.55}$$

A similar procedure starting with $\boldsymbol{c}_k$ of $(1.53)_2$ gives

$$e_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k} - u_{m,k}u_{m,l}\right) \ . \tag{1.56}$$

We can see that both E_{KL} and e_{kl} are symmetric tensors, that is,

$$E_{KL} = E_{LK} \ , \qquad e_{kl} = e_{lk} \ . \tag{1.57}$$

Therefore, in three dimensions there are only six independent components for each of these tensors, for example, E_{11}, E_{22}, E_{33}, $E_{12} = E_{21}$, $E_{13} = E_{31}$, and $E_{23} = E_{32}$. The first three components E_{11}, E_{22}, and E_{33} are called *normal strains* and the last three E_{12}, E_{13}, and E_{23} are called *shear strains.* The reason for this will be discussed later in this chapter.

1.6 Geometric linearization

The whole kinematics of deformable bodies is considerably simplified, if deformations are assumed to be small. In this context, a convenient measure of smallness is the norm of the displacement gradient,

$$\delta = \|\boldsymbol{H}\| = \sqrt{\boldsymbol{H} : \boldsymbol{H}} = \sqrt{U_{K,L}U_{L,K}} \ . \tag{1.58}$$

In the following, the term small deformation will correspond to the case of small displacement gradients. This means that all first derivatives of the displacements with respect to the coordinates are sufficiently small such that linearization is justified, that is, all terms of higher order are ignored. In its geometrical interpretation a small value of δ implies small strains as well as small rotations.

In the context of small strains and rotations *geometrical linearization* is the process of developing of all kinematics variables with respect to δ and dropping all terms of orders higher than $O(\delta)$. This definition implies the following asymptotic relations:

$$\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{H}^T \ , \qquad \text{(exact)} \tag{1.59}$$

$$\boldsymbol{F}^{-1} = \boldsymbol{I} - \boldsymbol{H}^T + O(\delta^2) \ , \tag{1.60}$$

$$\det\boldsymbol{F} = 1 + \operatorname{tr}\boldsymbol{H} + O(\delta^2) \ , \tag{1.61}$$

$$\boldsymbol{C} = \boldsymbol{I} + \boldsymbol{H} + \boldsymbol{H}^T + O(\delta^2) \ , \tag{1.62}$$

$$\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{H} - \boldsymbol{H}^T + O(\delta^2) \ , \tag{1.63}$$

$$\boldsymbol{U} = \boldsymbol{I} + \frac{1}{2}\left(\boldsymbol{H} + \boldsymbol{H}^T\right) + O(\delta^2)\ , \tag{1.64}$$

$$\boldsymbol{V} = \boldsymbol{I} + \frac{1}{2}\left(\boldsymbol{H} + \boldsymbol{H}^T\right) + O(\delta^2)\ , \tag{1.65}$$

$$\boldsymbol{R} = \boldsymbol{I} + \frac{1}{2}\left(\boldsymbol{H}^T - \boldsymbol{H}\right) + O(\delta^2)\ . \tag{1.66}$$

From the last three relations we infer

$$\begin{aligned}\boldsymbol{R}\cdot\boldsymbol{U} &= \left[\boldsymbol{I} + \frac{1}{2}\left(\boldsymbol{H}^T - \boldsymbol{H}\right) + O(\delta^2)\right]\cdot\left[\boldsymbol{I} + \frac{1}{2}\left(\boldsymbol{H} + \boldsymbol{H}^T\right) + O(\delta^2)\right] \\ &= \boldsymbol{I} + \boldsymbol{H}^T + O(\delta^2)\ ,\end{aligned} \tag{1.67}$$

and

$$\begin{aligned}\boldsymbol{V}\cdot\boldsymbol{R} &= \left[\boldsymbol{I} + \frac{1}{2}\left(\boldsymbol{H} + \boldsymbol{H}^T\right) + O(\delta^2)\right]\cdot\left[\boldsymbol{I} + \frac{1}{2}\left(\boldsymbol{H}^T - \boldsymbol{H}\right) + O(\delta^2)\right] \\ &= \boldsymbol{I} + \boldsymbol{H}^T + O(\delta^2)\ .\end{aligned} \tag{1.68}$$

Thus we can see that for small deformations the multiplicative decomposition of the deformation gradient into orthogonal, symmetric and positive definite factors is approximated by the additive decomposition of the displacement gradient into symmetric and skew-symmetric parts:

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{R}\cdot\boldsymbol{U} = \boldsymbol{V}\cdot\boldsymbol{R} = \boldsymbol{I} + \boldsymbol{H}^T = \\ &= \boldsymbol{I} + \frac{1}{2}\left(\boldsymbol{H} + \boldsymbol{H}^T\right) + \frac{1}{2}\left(\boldsymbol{H}^T - \boldsymbol{H}\right) = \\ &= \boldsymbol{I} + \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{R}}\end{aligned} \tag{1.69}$$

The symmetric part of the displacement gradient,

$$\tilde{\boldsymbol{E}} = \frac{1}{2}\left(\boldsymbol{H} + \boldsymbol{H}^T\right) \qquad \text{or} \qquad \tilde{E}_{KL} := \frac{1}{2}\left(U_{K,L} + U_{L,K}\right)\ , \tag{1.70}$$

is called the *linearized* or *infinitesimal strain tensor*, and the skew-symmetric part

$$\tilde{\boldsymbol{R}} := \frac{1}{2}\left(\boldsymbol{H}^T - \boldsymbol{H}\right) \qquad \text{or} \qquad \tilde{R}_{KL} := \frac{1}{2}\left(U_{K,L} - U_{L,K}\right)\ , \tag{1.71}$$

is called the *linearized* or *infinitesimal rotation tensor*. Summing up the last two equations results in

$$\boldsymbol{H}^T = \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{R}} \qquad \text{or} \qquad U_{K,L} = \tilde{E}_{KL} + \tilde{R}_{KL}\ . \tag{1.72}$$

Carrying this into (1.55), we obtain

$$\boldsymbol{E} = \tilde{\boldsymbol{E}} + \frac{1}{2}\left(\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{R}}\right)^T\cdot\left(\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{R}}\right)\ . \tag{1.73}$$

Now, it is clear that in order that $\boldsymbol{E} \approx \tilde{\boldsymbol{E}}$, not only strains $\tilde{\boldsymbol{E}}$ must be small, but rotations $\tilde{\boldsymbol{R}}$ must also be small so that products such as $\tilde{\boldsymbol{E}}^T \cdot \tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}^T \cdot \tilde{\boldsymbol{R}}$, and $\tilde{\boldsymbol{R}}^T \cdot \tilde{\boldsymbol{R}}$ will be negligible compared to $\tilde{\boldsymbol{E}}$.

A similar procedure for the Eulerian strain tensor e_{kl} gives

$$e_{kl} = \tilde{e}_{kl} - \frac{1}{2}\left(\tilde{e}_{mk} + \tilde{r}_{mk}\right)\left(\tilde{e}_{ml} + \tilde{r}_{ml}\right) \ , \tag{1.74}$$

where the infinitesimal strain tensor $\tilde{e}_{kl}$ and infinitesimal rotation tensor $\tilde{r}_{kl}$ are defined by

$$\tilde{e}_{kl} := \frac{1}{2}\left(u_{k,l} + u_{l,k}\right) \ , \qquad \tilde{r}_{kl} := \frac{1}{2}\left(u_{k,l} - u_{l,k}\right) \ . \tag{1.75}$$

The infinitesimal strains are again symmetric, and the infinitesimal rotations are skew-symmetric, that is,

$$\tilde{e}_{kl} = \tilde{e}_{lk} \ , \qquad \tilde{r}_{kl} = -\tilde{r}_{lk} \ . \tag{1.76}$$

When all displacement gradients $u_{k,l}$ are much small as compared to the unity, $e_{kl} \approx \tilde{e}_{kl}$. In the *linear theory (the infinitesimal deformation theory)* we assume that $E_{KL} = \tilde{E}_{KL}$ and $e_{kl} = \tilde{e}_{kl}$. In this case, (1.47) gives upon linearization

$$\tilde{E}_{KL} = \tilde{e}_{kl}\delta_{kK}\delta_{lL} \ , \qquad \tilde{e}_{kl} = \tilde{E}_{KL}\delta_{Kk}\delta_{Ll} \ , \tag{1.77}$$

where $\delta_{Kk} = \delta_{kK}$ is now the Kronecker delta. Thus, in the *linear theory the distinction between the Lagrangian and Eulerian strain tensors disappears.*

1.7 Length and angle changes

A geometrical meaning for strains and rotations is provided by considering the length and angle changes as a result of the deformation. Referred to the same rectangular coordinates X_K, an infinitesimal rectangular parallelepiped having edge vectors $\boldsymbol{I}_1 dX_1$, $\boldsymbol{I}_2 dX_2$, and $\boldsymbol{I}_3 dX_3$ at $\boldsymbol{X}$ after deformation becomes a rectilinear parallelepiped at $\boldsymbol{x}$ with corresponding edge vectors $\boldsymbol{C}_1 dX_1$, $\boldsymbol{C}_2 dX_2$, and $\boldsymbol{C}_3 dX_3$ (Figure 1.4). That is, a vector $d\boldsymbol{X}$ at $\boldsymbol{X}$ after deformation becomes $d\boldsymbol{x}$ at $\boldsymbol{x}$:

$$d\boldsymbol{X} = \boldsymbol{I}_K dX_K \ , \qquad d\boldsymbol{x} = \boldsymbol{C}_K dX_K \ . \tag{1.78}$$

If $\boldsymbol{N}$ and $\boldsymbol{n}$ are, respectively, unit vectors along $d\boldsymbol{X}$ and $d\boldsymbol{x}$, we have

$$N_K := \frac{dX_K}{|d\boldsymbol{X}|} = \frac{dX_K}{dS} \ , \qquad n_k := \frac{dx_k}{|d\boldsymbol{x}|} = \frac{dx_k}{ds} \ , \tag{1.79}$$

where dS and ds are length of $d\boldsymbol{X}$ and $d\boldsymbol{x}$, respectively. The ratio ds/dS of the lengths of $d\boldsymbol{x}$ and $d\boldsymbol{X}$ is called the *stretch.* This ratio may be expressed in terms of either $\boldsymbol{N}$ or $\boldsymbol{n}$. To indicate these dependence, we denote the stretch either by $\Lambda_{(\boldsymbol{N})}$ or by $\lambda_{(\boldsymbol{n})}$. They are, of course, the same physical quantity expressed differently, that is $\Lambda_{(\boldsymbol{N})} = \lambda_{(\boldsymbol{n})}$:

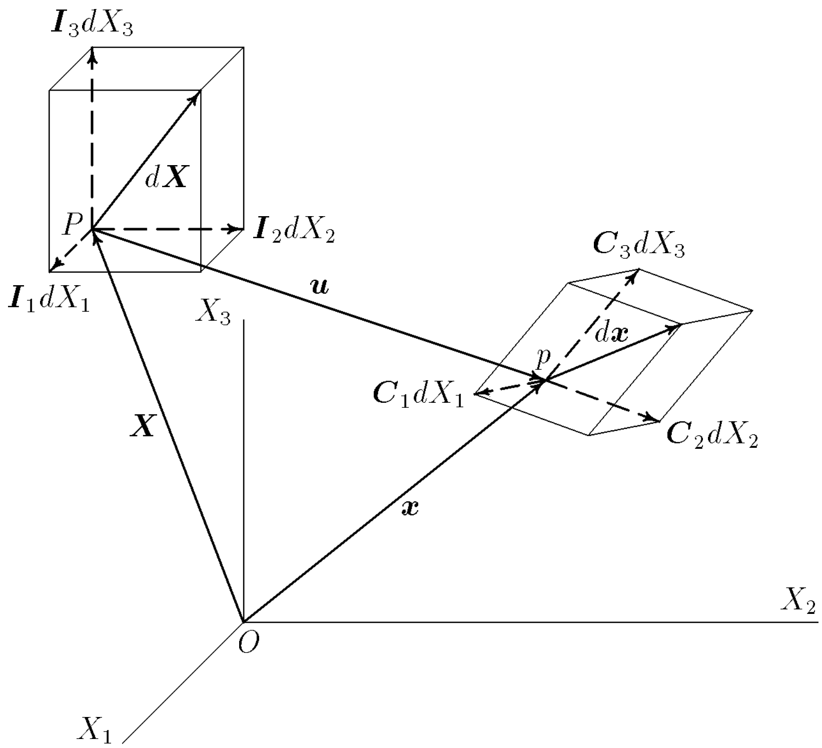

Figure 1.4. Deformation of an infinitesimal rectangular parallelepiped.

$$\Lambda_{(\boldsymbol{N})} := \frac{ds}{dS} = \sqrt{C_{KL} N_K N_L}\ , \qquad \lambda_{(\boldsymbol{n})} := \frac{ds}{dS} = \frac{1}{\sqrt{c_{kl} n_k n_l}}\ . \tag{1.80}$$

From this it is clear that the *normal components of* $\boldsymbol{C}$ *and* $\boldsymbol{c}$ *in the directions of* $\boldsymbol{N}$ *and* $\boldsymbol{n}$ *are, respectively, the squares and the inverse squares of stretches in these directions.* This point is clarified further if we select $\boldsymbol{N}$, for example, along the X_1-axis. Then $N_1 = 1$, $N_2 = N_3 = 0$, and $(1.80)_1$ gives

$$\Lambda_1 = \sqrt{C_{11}} = \sqrt{1 + E_{11}}\ . \tag{1.81}$$

The *extension* $E_{(\boldsymbol{N})} = e_{(\boldsymbol{n})}$ is defined by

$$E_{(\boldsymbol{N})} := \Lambda_{(\boldsymbol{N})} - 1 = \frac{ds - dS}{dS}\ . \tag{1.82}$$

Dividing (1.45) by dS^2 and using $(1.79)_1$ we have

$$\frac{ds^2 - dS^2}{dS^2} = 2E_{KL} N_K N_L\ . \tag{1.83}$$

Expressing the left-hand side of (1.83) by $E_{(\boldsymbol{N})}$, we get the quadratic equation for $E_{(\boldsymbol{N})}$:

$$E_{(\boldsymbol{N})}\left(E_{(\boldsymbol{N})} + 2\right) - 2E_{KL} N_K N_L = 0\ . \tag{1.84}$$

From the two solutions of this equation, we choose physically admissible one:

$$E_{(\boldsymbol{N})} = -1 + \sqrt{1 + 2E_{KL}N_K N_L}\ . \tag{1.85}$$

Particularly, when $\boldsymbol{N}$ is taken along the X_1-axis, this gives

$$E_{(1)} = -1 + \sqrt{1 + 2E_{11}}\ . \tag{1.86}$$

When the deformation is small, $E_{11} \ll 1$, by expanding (1.86) and neglecting the square and higher powers of E_{11}, we get

$$E_{11} \approx E_{(1)} \approx \tilde{E}_{11}\ . \tag{1.87}$$

Similar results are of course valid for E_{22} and E_{33}, which indicates that the *infinitesimal normal strains are approximately the extensions of the fibers along the coordinate axes when the deformation is small.*

The geometrical meaning of the shear strains E_{12}, E_{13}, and E_{23} is found by considering the angles between two directions $\boldsymbol{N}^{(1)}$ and $\boldsymbol{N}^{(2)}$,

$$\boldsymbol{N}^{(1)} = \frac{d\boldsymbol{X}^{(1)}}{dS^{(1)}}\ , \qquad \boldsymbol{N}^{(2)} = \frac{d\boldsymbol{X}^{(2)}}{dS^{(2)}}\ . \tag{1.88}$$

The angle Θ between these vectors in the undeformed body,

$$\cos\Theta = \frac{d\boldsymbol{X}^{(1)}}{dS^{(1)}} \cdot \frac{d\boldsymbol{X}^{(2)}}{dS^{(2)}}\ , \tag{1.89}$$

is changed by deformation to

$$\cos\theta = \frac{d\boldsymbol{x}^{(1)}}{ds^{(1)}} \cdot \frac{d\boldsymbol{x}^{(2)}}{ds^{(2)}} = \boldsymbol{C}_K \cdot \boldsymbol{C}_L \frac{dX_K^{(1)}}{ds^{(1)}} \frac{dX_L^{(2)}}{ds^{(2)}}\ , \tag{1.90}$$

By (1.30) and (1.88) we further get

$$\cos\theta = C_{KL} N_K^{(1)} N_L^{(2)} \frac{dS^{(1)}}{ds^{(1)}} \frac{dS^{(2)}}{ds^{(2)}} = \frac{C_{KL} N_K^{(1)} N_L^{(2)}}{\left(E_{(\boldsymbol{N}_1)} + 1\right)\left(E_{(\boldsymbol{N}_2)} + 1\right)}\ , \tag{1.91}$$

which can be rewritten in terms of the Lagrangian strain tensor as

$$\cos\theta = \frac{\left(2E_{KL} + \delta_{KL}\right) N_K^{(1)} N_L^{(2)}}{\sqrt{1 + 2E_{MN} N_M^{(1)} N_N^{(1)}}\sqrt{1 + 2E_{RS} N_R^{(2)} N_S^{(2)}}}\ . \tag{1.92}$$

When $\boldsymbol{N}^{(1)}$ is taken along X_1-axis and $\boldsymbol{N}^{(2)}$ along X_2-axis, (1.92) reduces to

$$\cos\theta_{(12)} = \frac{2E_{12}}{\sqrt{1 + 2E_{11}}\sqrt{1 + 2E_{22}}}\ . \tag{1.93}$$

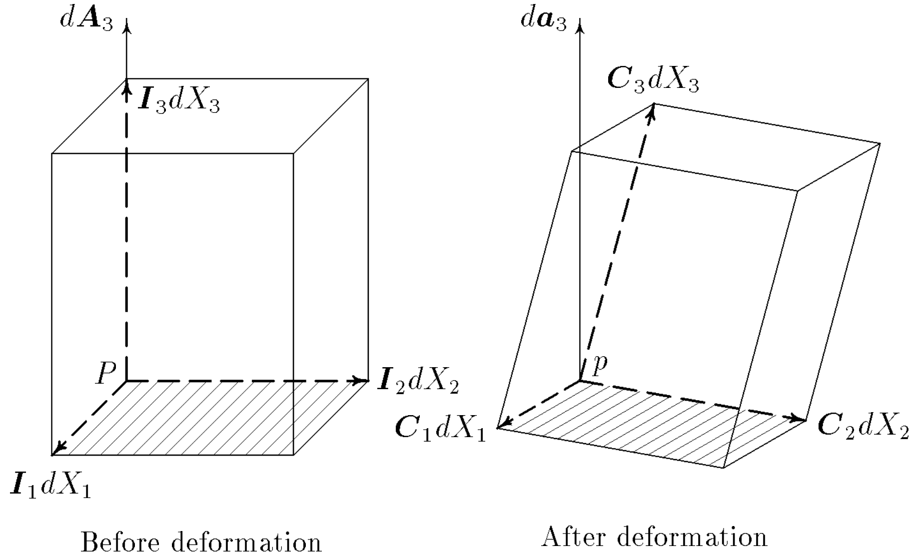

Figure 1.5. Deformation of an infinitesimal rectangular parallelepiped.

When the components of the Lagrangian strain tensor are small as compared to unity, we have approximately

$$2E_{12} \approx 2\tilde{E}_{12} \approx \cos\theta_{(12)} \ . \tag{1.94}$$

Hence, writing $\cos\theta_{(12)} = \sin\Gamma_{(12)} \approx \Gamma_{(12)}$, we have

$$2E_{12} \approx 2\tilde{E}_{12} \approx \Gamma_{(12)} \ . \tag{1.95}$$

Similar results are valid for E_{13} and E_{23}. This provides geometrical meaning for shear strains. *The infinitesimal shear strains are approximately one half of the angle change between the coordinate axes for small deformations.*

1.8 Area and volume changes

Let us investigate the change of area and volume with deformation. We have already found that (cf. Section 1.7) an infinitesimal rectangular parallelepiped with edge vectors $\boldsymbol{I}_1 dX_1$, $\boldsymbol{I}_2 dX_2$, and $\boldsymbol{I}_3 dX_3$ after deformation becomes a rectilinear parallelepiped with edge vectors $\boldsymbol{C}_1 dX_1$, $\boldsymbol{C}_2 dX_2$, and $\boldsymbol{C}_3 dX_3$ (Figure 1.5). Thus the deformed area is given by

$$d\boldsymbol{a}_3 = \boldsymbol{C}_1 dX_1 \times \boldsymbol{C}_2 dX_2 = x_{k,1} x_{l,2} \boldsymbol{i}_k \times \boldsymbol{i}_l dX_1 dX_2 \ , \tag{1.96}$$

or since $dA_3 = dX_1 dX_2$ and $\boldsymbol{i}_k \times \boldsymbol{i}_l = \epsilon_{klm} \boldsymbol{i}_m$, this reads

$$d\boldsymbol{a}_3 = x_{k,1} x_{l,2} \epsilon_{klm} \boldsymbol{i}_m dA_3 \ . \tag{1.97}$$

But from Jacobi's identity (1.25) we have

$$j X_{3,m} = \epsilon_{klm} x_{k,1} x_{l,2} \ , \tag{1.98}$$

so that

$$d\boldsymbol{a}_3 = jX_{3,m}dA_3\boldsymbol{i}_m \ . \tag{1.99}$$

Analogous expressions are valid for $d\boldsymbol{a}_1$ and $d\boldsymbol{a}_2$. Thus

$$d\boldsymbol{a} = d\boldsymbol{a}_1 + d\boldsymbol{a}_2 + d\boldsymbol{a}_3 = jX_{K,k}dA_K\boldsymbol{i}_k \ , \tag{1.100}$$

whose kth component is

$$da_k = jX_{K,k}dA_K \ . \tag{1.101}$$

To calculate the deformed volume element, we take the scalar product of $d\boldsymbol{a}_3$ with $\boldsymbol{C}_3dX_3$:

$$dv = d\boldsymbol{a}_3 \cdot \boldsymbol{C}_3dX_3 = jX_{3,k}\boldsymbol{i}_k \cdot (x_{m,3}\boldsymbol{i}_m)\, dA_3dX_3 = jX_{3,k}x_{m,3}\delta_{km}dV \ . \tag{1.102}$$

Hence

$$dv = jdV \ . \tag{1.103}$$

1.9 Change of the unit normal

To write the jump conditions in the reference frame, we need to find the relation between unit exterior normals $\boldsymbol{n} = n_k\boldsymbol{i}_k$ and $\boldsymbol{N} = N_K\boldsymbol{I}_K$ of the deformed surface $s(t)$ and the undeformed surface S, respectively. From (1.101) we have

$$da_k = jX_{K,k}dA_K \ , \tag{1.104}$$

but

$$n_k = \frac{da_k}{\sqrt{da_l da_l}} = \frac{da_k}{da} \ , \qquad N_K = \frac{dA_K}{\sqrt{dA_L dA_L}} = \frac{dA_K}{dA} \ . \tag{1.105}$$

Hence

$$n_k = jX_{L,k}N_L\frac{dA}{da} \ . \tag{1.106}$$

Using (1.104), we have

$$da = \sqrt{da_l da_l} = j\sqrt{X_{K,l}X_{L,l}dA_K dA_L} = j\sqrt{X_{K,l}X_{L,l}N_K N_L}dA \ , \tag{1.107}$$

which gives

$$\frac{da}{dA} = j\sqrt{B_{KL}N_K N_L} = j\sqrt{\boldsymbol{N}\cdot\boldsymbol{B}\cdot\boldsymbol{N}} \ , \tag{1.108}$$

where $\boldsymbol{B}$ is the Piola deformation tensor defined by $(1.32)_2$. Substituting (1.108) into (1.106), we get

$$n_k = \frac{X_{L,k}N_L}{\sqrt{B_{MN}N_M N_N}} \quad \text{or} \quad \boldsymbol{n} = \frac{\boldsymbol{N}\cdot\boldsymbol{F}^{-1}}{\sqrt{\boldsymbol{N}\cdot\boldsymbol{B}\cdot\boldsymbol{N}}} \ . \tag{1.109}$$

Let us express the foregoing expressions in the linear theory in which the displacement gradient is sufficiently small that the linearization is justified. Using the linearized form (1.63) of the Piola deformation tensor $\boldsymbol{B}$, we can write

$$\frac{1}{\sqrt{\boldsymbol{N}\cdot\boldsymbol{B}\cdot\boldsymbol{N}}} = \frac{1}{\sqrt{1-2\boldsymbol{N}\cdot\boldsymbol{H}\cdot\boldsymbol{N}}} = 1 + \boldsymbol{N}\cdot\boldsymbol{H}\cdot\boldsymbol{N} + O\left(\delta^2\right) \ .$$

Employing this and the linearized forms (1.60) and (1.61) for the spatial deformation gradient $\boldsymbol{F}^{-1}$ and the jacobian j, the unit normal $\boldsymbol{n}$ to the deformed surface $s(t)$ and the ratio dA/da may, within the framework of linear approximation, be written as

$$\boldsymbol{n} = (1 + \boldsymbol{N}\cdot\boldsymbol{H}\cdot\boldsymbol{N})\boldsymbol{N} - \boldsymbol{H}\cdot\boldsymbol{N} + O\left(\delta^2\right) \ , \tag{1.110}$$

$$\frac{dA}{da} = 1 + \boldsymbol{N}\cdot\boldsymbol{H}\cdot\boldsymbol{N} - \operatorname{tr}\boldsymbol{H} + O\left(\delta^2\right) \ . \tag{1.111}$$

2. KINEMATICS

2.1 Material and spatial time derivatives

If we focus attention on a specific particle X^P having the material position vector $\boldsymbol{X}^P$, (1.3) takes the form

$$\boldsymbol{x}^P = \boldsymbol{\chi}(\boldsymbol{X}^P, t) \tag{2.1}$$

and describes the *path* or *trajectory* of that particle as a function of time. The *velocity* $\boldsymbol{v}^P$ of the particle along this its path is defined as the time rate of change of position, or

$$\boldsymbol{V}^P := \frac{d\boldsymbol{x}^P}{dt} = \left(\frac{\partial \boldsymbol{\chi}}{\partial t}\right)\Big|_{\boldsymbol{X}=\boldsymbol{X}^P} , \tag{2.2}$$

where the subscript $\boldsymbol{X}$ accompanying a vertical bar indicates that $\boldsymbol{X}$ is held constant (equal to $\boldsymbol{X}^P$) in the differentiation of $\boldsymbol{\chi}$. In an obvious generalization, we may define the *velocity* of the total body as the derivative

$$\boldsymbol{V}(\boldsymbol{X}, t) := \frac{d\boldsymbol{x}}{dt} = \left(\frac{\partial \boldsymbol{\chi}}{\partial t}\right)\Big|_{\boldsymbol{X}} . \tag{2.3}$$

This is the Lagrangian representation of velocity and the time rate of change with respect to a *moving* particle is called the *material derivative*. Similarly, the material derivative of $\boldsymbol{v}$ defines the Lagrangian representation of acceleration,

$$\boldsymbol{A}(\boldsymbol{X}, t) := \frac{d\boldsymbol{v}}{dt} = \left(\frac{\partial \boldsymbol{V}}{\partial t}\right)\Big|_{\boldsymbol{X}} = \left(\frac{\partial^2 \boldsymbol{\chi}(\boldsymbol{X}, t)}{\partial^2 t}\right)\Big|_{\boldsymbol{X}} . \tag{2.4}$$

By using (1.48) we may also write

$$\boldsymbol{V}(\boldsymbol{X}, t) = \left(\frac{\partial \boldsymbol{U}}{\partial t}\right)\Big|_{\boldsymbol{X}} , \qquad \boldsymbol{A}(\boldsymbol{X}, t) = \left(\frac{\partial^2 \boldsymbol{U}}{\partial^2 t}\right)\Big|_{\boldsymbol{X}} . \tag{2.5}$$

In the Lagrangian representation, the material particle with a given velocity or acceleration is identifiable, whereas, in the Eulerian description, the velocity and acceleration at time t at a spatial point are known, but the particle occupying this point is not known, so that, $\boldsymbol{\chi}(\boldsymbol{X}, t)$ cannot be specified. This excludes the knowledge of the material derivative of $\boldsymbol{\chi}$.

Since the fundamental laws of continuum dynamics involve the acceleration of particles and since the Lagrangian formulation of velocity may not be available, the acceleration must be calculated from the Eulerian formulation of velocity. To accomplish this, only the *existence* of the unknown trajectories, $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$, must be assumed. Substitution of $(1.3)_2$ for $\boldsymbol{X}$ in (2.3), we have

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{V}(\boldsymbol{\chi}^{-1}(\boldsymbol{x}, t), t) , \tag{2.6}$$

which gives the velocity field at each spatial point $\boldsymbol{x}$ at time t with no specification of its relation to the material point $\boldsymbol{X}$. This is the Eulerian representation of velocity.

Based on the same assumption of the existence of the trajectories $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$, the Eulerian representation of velocity (2.6) may be considered in the form $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}(\boldsymbol{X}, t), t)$. By chain rule of calculus, we get

$$\left.\frac{d\boldsymbol{v}}{dt}\right|_{\boldsymbol{X}} = \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v} \cdot \operatorname{grad} \boldsymbol{v} \ , \tag{2.7}$$

where differentiation in the grad-operator is taken with respect to the spatial variables. This is the desired equation for the Eulerian representation of acceleration which is expressed in terms of the Eulerian representation of velocity,

$$\boldsymbol{a}(\boldsymbol{x}, t) = \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v} \cdot \operatorname{grad} \boldsymbol{v} \ . \tag{2.8}$$

In this equation, the first term on the right-hand side gives the time rate of change of velocity at a fixed position $\boldsymbol{x}$, known as the *local rate of change* or *spatial time derivatives*; the second term results from the particles changing position in space and is referred to as the *convective term*.

The material derivative of any other field quantity can be calculated in the same way if its Lagrangian or Eulerian representation is known. This suggests to introduce the *material derivative operator*

$$\frac{D}{Dt} := \left.\frac{d}{dt}\right|_{\boldsymbol{X}} = \begin{cases} \left.\dfrac{\partial}{\partial t}\right|_{\boldsymbol{X}} & \text{for a field in the Lagrangian representation ,} \\[2ex] \left.\dfrac{\partial}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v} \cdot \operatorname{grad} & \text{for a field in the Eulerian representation ,} \end{cases} \tag{2.9}$$

which can be applied to any field quantity given in the Lagrangian or Eulerian representation.

2.2 Reynolds's transport theorem

The material derivative of surface and volume integrals are required in the formulation of fundamental laws of continuum mechanics. To provide necessary apparatus we now give

Fundamental Lemma. *The material derivative of the deformation gradients is given by*

$$\frac{D}{Dt}(x_{k,K}) = v_{k,l} x_{l,K} \ . \tag{2.10}$$

The proof is immediate, for

$$\frac{D}{Dt}(x_{k,K}) = \frac{D}{Dt}\left(\frac{\partial x_k}{\partial X_K}\right) = \frac{\partial}{\partial X_K}\left(\frac{D x_k}{Dt}\right) = v_{k,K} = v_{k,l} x_{l,K} \ ,$$

since in the operation D/Dt we have X_K fixed so that D/Dt and $\partial/\partial X_K$ commute.

A corollary of this lemma is

$$\frac{D}{Dt}\left(X_{K,k}\right) = -v_{l,k}X_{K,l} \ . \tag{2.11}$$

To prove it, we take the material derivative of $x_{l,L}X_{L,k} = \delta_{kl}$. Hence

$$\frac{D}{Dt}\left(x_{l,L}\right)X_{L,k} + x_{l,L}\frac{D}{Dt}\left(X_{L,k}\right) = 0 \ .$$

Using (2.10), we find that this expression reads

$$x_{l,L}\frac{D}{Dt}\left(X_{L,k}\right) = -v_{l,m}x_{m,L}X_{L,k} = -v_{l,k} \ .$$

Multiplying both sides by $X_{K,l}$, we obtain (2.11).

LEMMA 1: *The material derivative of the Lagrangian strain tensor is given by*

$$\frac{D}{Dt}(E_{KL}) = \frac{1}{2}\frac{D}{Dt}(C_{KL}) = d_{kl}x_{k,K}x_{l,L} \ , \tag{2.12}$$

where

$$d_{kl} := \frac{1}{2}\left(v_{k,l} + v_{l,k}\right) \qquad \text{or} \qquad \boldsymbol{d} := \frac{1}{2}\left(\operatorname{grad}\boldsymbol{v} + (\operatorname{grad}\boldsymbol{v})^{\mathrm{T}}\right) \tag{2.13}$$

is the symmetric part of grad $\boldsymbol{v}$, often called the *strain-rate tensor.* To show this, we have

$$\frac{D}{Dt}(C_{KL}) = \frac{D}{Dt}\left(x_{k,K}x_{k,L}\right) = v_{k,l}x_{l,K}x_{k,L} + x_{k,K}v_{k,l}x_{l,L} = \left(v_{k,l} + v_{l,k}\right)x_{k,K}x_{l,L} = 2d_{kl}x_{k,K}x_{l,L} \ .$$

LEMMA 2: *The material derivative of the jacobian is given by*

$$\frac{Dj}{Dt} = j \operatorname{div}\boldsymbol{v} \ . \tag{2.14}$$

To show this, we have

$$\frac{Dj}{Dt} = \frac{D}{Dt}\left(\det x_{k,K}\right) = \frac{\partial j}{\partial x_{k,K}}\frac{D(x_{k,K})}{Dt} = \frac{\partial j}{\partial x_{k,K}}v_{k,l}x_{l,K} \ ,$$

Using (1.25), this gives (2.14).

THEOREM: *The material derivative of a volume integral of any scalar or vector field ϕ over the spatial volume $v(t)$ is given by*

$$\frac{D}{Dt}\int_{v(t)}\phi dv = \int_{v(t)}\left(\frac{D\phi}{Dt} + \phi \operatorname{div}\boldsymbol{v}\right)dv \ . \tag{2.15}$$

Proof: Under the assumption of existence of the mapping (1.3), we firstly transform the integral over the spatial volume to an integral over the material volume V. By (2.14), we have

$$\frac{D}{Dt}\int_{v(t)}\phi dv = \frac{D}{Dt}\int_{V}\Phi j dV \ ,$$

where $\Phi(\boldsymbol{X}, t) = \phi(\boldsymbol{x}(\boldsymbol{X}, t), t)$. Since V is a fixed volume in the Lagrangian configuration, the differentiation D/Dt and the integration over V commute and the differentiation D/Dt can be performed inside the integral sign,

$$\frac{D}{Dt}\int_V \Phi j dV = \int_V \frac{D}{Dt}(\Phi j)\, dV = \int_V \left(\frac{D\Phi}{Dt} j + \Phi \frac{Dj}{Dt}\right) dV = \int_V \left(\frac{D\Phi}{Dt} + \Phi \text{div}\, \boldsymbol{v}\right) j dV \ .$$

By converting this back to the spatial formulation by (2.14), we prove (2.15). Equation (2.15) is often spoken of as the *Reynolds transport theorem.*

This theorem may be expressed in a different form. To do it, we calculate the material derivative of a field ϕ,

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \boldsymbol{v} \cdot \text{grad}\, \phi \ , \tag{2.16}$$

and substitute this back into (2.15). With the product rule

$$\phi\, \text{div}\, \boldsymbol{v} + \boldsymbol{v} \cdot \text{grad}\, \phi = \text{div}\, (\boldsymbol{v}\phi) \tag{2.17}$$

which is valid for a scalar or vector field ϕ, we then arrive

$$\frac{D}{Dt}\int_{v(t)} \phi dv = \int_{v(t)} \left(\frac{\partial \phi}{\partial t} + \text{div}\, (\boldsymbol{v}\phi)\right) dv \ .$$

Arranging the second term on the right-hand side according to the generalized Gauss's theorem

$$\int_v \text{div}\, \boldsymbol{A}\ dv = \oint_s \boldsymbol{n} \cdot \boldsymbol{A}\ da \ , \tag{2.18}$$

where $\boldsymbol{A}$ is a second-order tensor-valued function continuously differentiable in v, s is the surface bounding volume v and $\boldsymbol{n}$ is the outward unit normal to s, we get an equivalent form of the Reynolds transport theorem

$$\frac{D}{Dt}\int_{v(t)} \phi dv = \int_{v(t)} \frac{\partial \phi}{\partial t} dv + \oint_{s(t)} \phi\, (\boldsymbol{n} \cdot \boldsymbol{v})\ da \ , \tag{2.19}$$

where both ϕ and $\boldsymbol{v}$ are again required to be continuously differentiable in v.

2.3 Modified Reynolds's transport theorem

The time rate of integral over a region containing a discontinuity surface is common occurrence in the study of geophysical phenomena. We give below an expression modifying the Reynolds transport theorem for the case when a volume is intersected by a moving discontinuity surface.

Consider a material volume v which is intersected by a discontinuity surface $\sigma(t)$ across which a tensor-valued function $\boldsymbol{A}$ undergoes a jump (Figure 2.1). The surface $\sigma(t)$ divides the material volume v into two parts, namely v^+ on the side of the normal $\boldsymbol{n}$ and v^- on the other side. Then the generalized Gauss theorem (2.18) is to be modified as

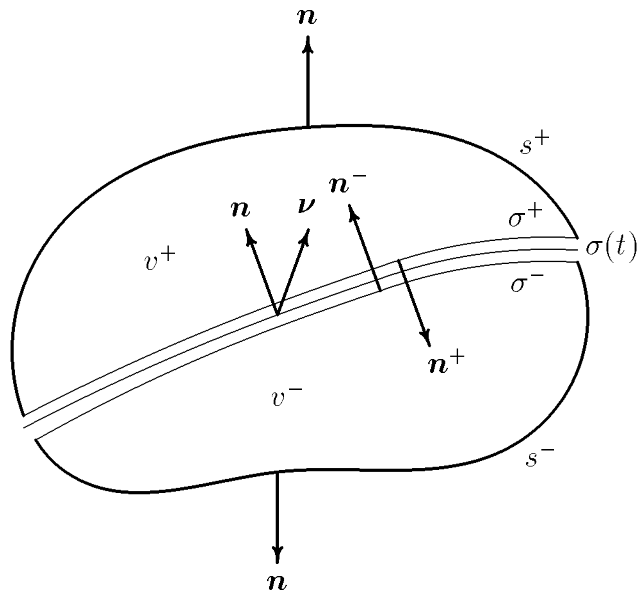

Figure 2.1. Discontinuity surface.

$$\int_{v-\sigma} \operatorname{div} \boldsymbol{A} \; dv = \oint_{s-\sigma} \boldsymbol{n} \cdot \boldsymbol{A} \; da - \int_{\sigma} \boldsymbol{n} \cdot [\boldsymbol{A}]_{-}^{+} \; da \; . \tag{2.20}$$

The volume integral over $v - \sigma$ means the volume v of the body excluding the material points located on the discontinuity surface σ. Similarly, the integral over the surface $s - \sigma$ excludes the line of intersection of σ with s, that is,

$$v - \sigma := v^{+} + v^{-} \; , \qquad s - \sigma := s^{+} + s^{-} \; . \tag{2.21}$$

The brackets indicate the jump of the enclosed quantity across $\sigma(t)$, e.g.,

$$[\boldsymbol{A}]_{-}^{+} := \boldsymbol{A}^{+} - \boldsymbol{A}^{-} \; . \tag{2.22}$$

To prove (2.20) we apply the Gauss theorem (2.18) to the two volumes v^{+} and v^{-} bounded by $s^{+} + \sigma^{+}$ and $s^{-} + \sigma^{-}$, respectively. Hence

$$\int_{v^{+}} \operatorname{div} \boldsymbol{A} \; dv = \int_{s^{+}} \boldsymbol{n} \cdot \boldsymbol{A} \; da + \int_{\sigma^{+}} \boldsymbol{n}^{+} \cdot \boldsymbol{A}^{+} \; da \; ,$$

$$\int_{v^{-}} \operatorname{div} \boldsymbol{A} \; dv = \int_{s^{-}} \boldsymbol{n} \cdot \boldsymbol{A} \; da + \int_{\sigma^{-}} \boldsymbol{n}^{-} \cdot \boldsymbol{A}^{-} \; da \; ,$$

where $\boldsymbol{n}^{+}$ and $\boldsymbol{n}^{-}$ are the exterior normals to σ^{+} and σ^{-}, respectively. Upon adding these two equations, we get

$$\int_{v^{+}+v^{-}} \operatorname{div} \boldsymbol{A} \; dv = \oint_{s^{+}+s^{-}} \boldsymbol{n} \cdot \boldsymbol{A} \; da + \int_{\sigma^{+}} \boldsymbol{n}^{+} \cdot \boldsymbol{A}^{+} \; da + \int_{\sigma^{-}} \boldsymbol{n}^{-} \cdot \boldsymbol{A}^{-} \; da \; .$$

But we have

$$\boldsymbol{n}^+ = -\boldsymbol{n}^- = -\boldsymbol{n} \ ,$$

and letting σ^+ and σ^- approach σ, so that

$$\int_{\sigma^+} \boldsymbol{n}^+ \cdot \boldsymbol{A}^+ \ da + \int_{\sigma^-} \boldsymbol{n}^- \cdot \boldsymbol{A}^- \ da = \int_\sigma \boldsymbol{n} \cdot \left(\boldsymbol{A}^- - \boldsymbol{A}^+ \right) \ da = -\int_\sigma \boldsymbol{n} \cdot [\boldsymbol{A}]_-^+ \ da$$

The Reynolds transport theorem (2.15) has also to be modified once the discontinuity surface $\sigma(t)$ moves with velocity $\boldsymbol{\nu}$ which differs, in general, from the material velocity $\boldsymbol{v}$. The modification of (2.15) reads

$$\frac{D}{Dt} \int_{v-\sigma} \phi dv = \int_{v-\sigma} \left(\frac{D\phi}{Dt} + \phi \operatorname{div} \boldsymbol{v} \right) dv + \int_\sigma [\phi(\boldsymbol{v} - \boldsymbol{\nu})]_-^+ \cdot \boldsymbol{n} \ da \ . \tag{2.23}$$

Both ϕ and $\boldsymbol{v}$ are required to be continuously differentiable in $v - \sigma$. To prove (2.23) we apply (2.19) to the two volumes v^+ and v^- bounded by $s^+ + \sigma^+$ and $s^- + \sigma^-$, respectively. Hence

$$\frac{D}{Dt} \int_{v^+} \phi dv = \int_{v^+} \frac{\partial \phi}{\partial t} dv + \int_{s^+} \phi \left(\boldsymbol{n} \cdot \boldsymbol{v} \right) \ da + \int_{\sigma^+} \phi \left(\boldsymbol{n} \cdot \boldsymbol{\nu} \right) \ da \ ,$$

$$\frac{D}{Dt} \int_{v^-} \phi dv = \int_{v^-} \frac{\partial \phi}{\partial t} dv + \int_{s^-} \phi \left(\boldsymbol{n} \cdot \boldsymbol{v} \right) \ da + \int_{\sigma^-} \phi \left(\boldsymbol{n} \cdot \boldsymbol{\nu} \right) \ da \ .$$

Upon adding these two equations, letting σ^+ and σ^- approach σ and realizing that $\boldsymbol{n}^+ = -\boldsymbol{n}^- = -\boldsymbol{n}$, we obtain

$$\frac{D}{Dt} \int_{v^+ + v^-} \phi dv = \int_{v^+ + v^-} \frac{\partial \phi}{\partial t} dv + \oint_{s^+ + s^-} \phi \left(\boldsymbol{n} \cdot \boldsymbol{v} \right) \ da - \int_\sigma [\phi \boldsymbol{\nu}]_-^+ \cdot \boldsymbol{n} \ da \ .$$

Using the Gauss theorem (2.20) for $\boldsymbol{A} = \boldsymbol{v}\phi$ to replace the second term on the right-hand side, we get

$$\frac{D}{Dt} \int_{v^+ + v^-} \phi dv = \int_{v^+ + v^-} \left(\frac{\partial \phi}{\partial t} + \operatorname{div} \left(\boldsymbol{v} \phi \right) \right) dv + \int_\sigma [\phi(\boldsymbol{v} - \boldsymbol{\nu})]_-^+ \cdot \boldsymbol{n} \ da \ .$$

To complete the proof of modified Reynolds's transport theorem (2.23), the first term on the right-hand side is to be arranged by making use of (2.16) and (2.17).

3. STRESS

3.1 Body and surface forces, mass density

The forces acting on a continuum or between portions of it divide into *long-range forces* and *short-range forces*.

Long-range forces comprise *gravitational, electromagnetic* and inertial forces. These forces decrease very gradually with an increase in distance between the interacting particles. As a result, long-range forces act uniformly on all matter contained in a sufficiently small volume, so that, they are proportional to its size. In continuum mechanics, long-range forces are therefore called *body* or *volume forces.*

Short-range forces comprise several types of *molecular* forces. Their characteristic feature is that they decrease extremely abruptly with an increase in distance between the interacting particles. Hence, they are appreciable only when this distance does not exceed molecular dimensions. A consequence is that, if the matter inside some volume is acted on by short-range forces originating from interactions with matter outside this volume, these forces can only act on a thin layer immediately below its surface. In continuum mechanics, short-range forces are therefore called *surface* forces; they are specified more closely by constitutive equations (Chapter 5).

In the following, we assume that volume and surface forces arise due to interactions that are *equal, opposite* and *collinear* (so-called the strong law of action and reaction). On this assumption, volume and surface couple stresses cannot arise.

In continuum mechanics, with each body there is associated a measure called *mass. It is non-negative and additive, and it is invariant under the motion.* If the mass is *absolutely continuous* in the space variables, then there exists a density ϱ called the *mass density.* The total mass of the body is then determined by

$$m = \int_v \varrho dv \ . \tag{3.1}$$

If the mass is not continuous throughout volume v, then instead of (3.1) we write

$$m = \int_{v_1} \varrho dv + \sum_\alpha m_\alpha \ , \tag{3.2}$$

where the summation is taken over all *discrete* masses contained in the body. We shall be dealing with a continuous mass medium in which (3.1) is valid, which implies that $m \to 0$ as $v \to 0$. We therefore have

$$0 \le \varrho < \infty \ . \tag{3.3}$$

Let $\boldsymbol{f}$ be the body force per unit mass. The resultant body force acting on the body currently occupying some finite volume v is then

$$\int_v \varrho \boldsymbol{f} dv \ .$$

3.2 Cauchy traction principle

We consider a material body $b(t)$ which is subject to body forces $\boldsymbol{f}$ and surface forces $\boldsymbol{g}$. Let p be an interior point of $b(t)$ and imagine a plane surface s^* passing through point p (sometimes referred to as a *cutting plane*) so as to partition the body into two portions, designated I and II (Figure 3.1). Point p is lying in the small element of area Δs^* of the cutting plane, which is defined by the unit normal $\boldsymbol{n}$ pointing in the direction from Portion I into Portion II, as shown in Figure 3.1. The internal forces being transmitted across the cutting plane due to the action of Portion II upon Portion I will give rise to a force distribution on Δs^* equivalent to a resultant surface force $\Delta \boldsymbol{g}$ as is also shown in Figure 3.1. (For simplicity, body forces and surface forces acting on the body as a whole are not drawn in Figure 3.1.) Notice that $\Delta \boldsymbol{g}$ are not necessarily in the direction of the unit normal vector $\boldsymbol{n}$. The *Cauchy traction principle* postulates that the limit when the area Δs^* shrinks to zero, with p remaining an interior point, exists and is

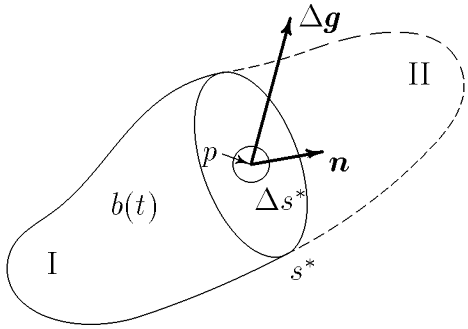

Figure 3.1.
Surface force on surface element Δs^*.

$$\boldsymbol{t}_{(\boldsymbol{n})} = \lim_{\Delta s^* \to 0} \frac{\Delta \boldsymbol{g}}{\Delta s^*} \,. \tag{3.4}$$

Obviously, this limit is meaningful only if Δs^* degenerates not into a curve but into a point p. The vector $\boldsymbol{t}_{(\boldsymbol{n})}$ is called the *Cauchy stress vector* or the *Cauchy traction vector* (force per unit area). It is important to note that, in general, $\boldsymbol{t}_{(\boldsymbol{n})}$ depends not only on the position of p on s but also the orientation of surface s, i.e., on its external normal $\boldsymbol{n}$. This dependence is therefore indicated by the subscript $\boldsymbol{n}$. Thus, for the infinity of cutting planes imaginable through point p, each identified by a specific $\boldsymbol{n}$, there is also an infinity of associated stress vectors $\boldsymbol{t}_{(\boldsymbol{n})}$ for a given loading of the body.

We incidentally mention that a continuous distribution of surface forces acting across some surface is, in general, equivalent to a resultant force and a resultant couple. In (3.4) we have made the assumption that, in the limit at p, the couple per unit area vanishes and therefore there is no remaining concentrated moment, or *couple stress* as it is called. For a discussion of couple stresses, the reader should referred to Eringen, 1967.

To determine the dependence of the stress vector on the exterior normal, we next apply the principle of balance of linear momentum to a small tetrahedron of volume Δv having its vertex at p, three coordinate surfaces Δa_k, and the base Δa on s with an oriented normal $\boldsymbol{n}$ (Figure 3.2). The stress vector [1] on the coordinate surface x_k = const. is denoted by $-\boldsymbol{t}_k$.

[1]Since the exterior normal of a coordinate surface x_k = const. is in the direction of $-x_k$, without loss in generality, we denote the stress vector acting on this coordinate surface by $-\boldsymbol{t}_k$ rather than $\boldsymbol{t}_k$.

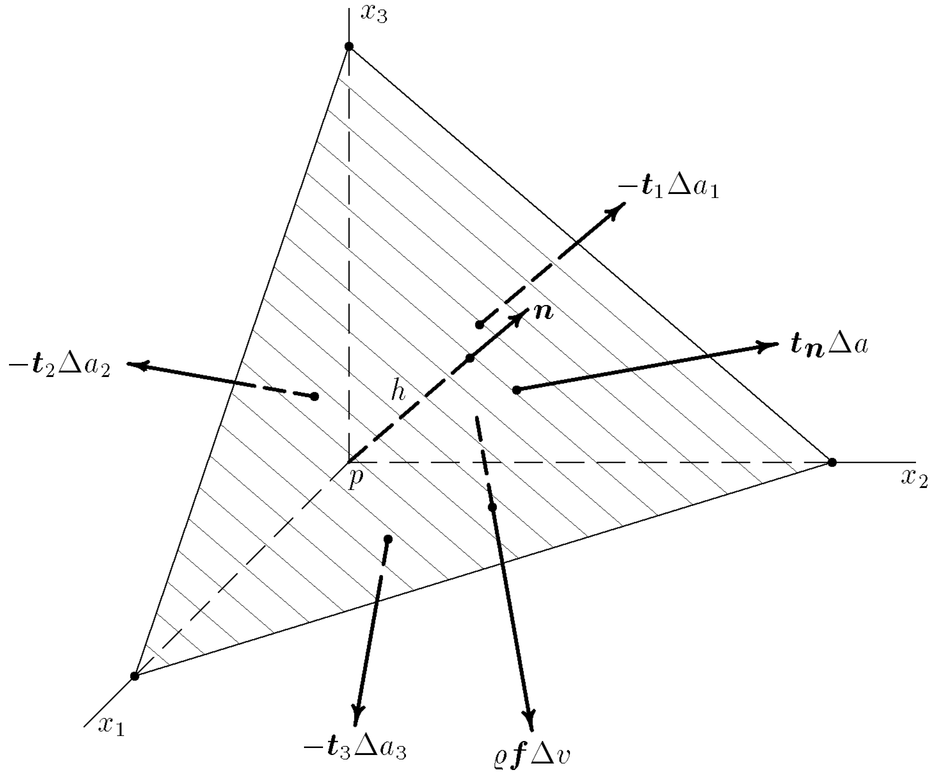

Figure 3.2. Equilibrium of an infinitesimal tetrahedron.

We now apply the equation of balance of linear momentum (Sect.4.1) to this tetrahedron.

$$\int_{\Delta v} \varrho \boldsymbol{f} dv - \int_{\Delta a_k} \boldsymbol{t}_k da_k + \int_{\Delta a} \boldsymbol{t}_{(\boldsymbol{n})} da = \frac{D}{Dt} \int_{\Delta v} \varrho \boldsymbol{v} dv \ .$$

An estimate of the surface and volume integrals may be made by use of the mean value theorem:

$$\varrho^* \boldsymbol{f}^* \Delta v - \boldsymbol{t}_k^* \Delta a_k + \boldsymbol{t}_{(\boldsymbol{n})}^* \Delta a = \frac{D}{Dt} \left(\varrho^* \boldsymbol{v}^* \Delta v \right) \ , \tag{3.5}$$

where ϱ^*, $\boldsymbol{f}^*$, and $\boldsymbol{v}^*$ are, respectively, the values of ϱ, $\boldsymbol{f}$, and $\boldsymbol{v}$ at some interior points of the tetrahedron and $\boldsymbol{t}_{(\boldsymbol{n})}^*$ and $\boldsymbol{t}_k^*$ are the values of $\boldsymbol{t}_{(\boldsymbol{n})}$ and $\boldsymbol{t}_k$ on the surface Δa and on coordinate surfaces Δa_k. The volume of the tetrahedron is given by

$$\Delta v = \frac{1}{3} h \Delta a \ , \tag{3.6}$$

where h is the perpendicular distance from point p to the base Δa. Moreover, the area vector $\Delta \boldsymbol{a}$ is equal to the sum of coordinate area vectors, i.e.,

$$\Delta \boldsymbol{a} = \boldsymbol{n} \Delta a = \Delta a_k \boldsymbol{i}_k \ . \tag{3.7}$$

Thus

$$\Delta a_k = n_k \Delta a \ . \tag{3.8}$$

Inserting (3.6) and (3.8) into (3.5) and canceling the common factor Δa, we obtain

$$\frac{1}{3}\varrho^* \boldsymbol{f}^* h - \boldsymbol{t}_k^* n_k + \boldsymbol{t}_{(\boldsymbol{n})}^* = \frac{1}{3}\frac{D}{Dt}\left(\varrho^* \boldsymbol{v}^* h\right) \ . \tag{3.9}$$

Now, letting the tetrahedron shrink to point p by taking the limit $h \to 0$ and noting that in this limiting process the starred quantities take on the actual values of those same quantities at point p, we have

$$\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{t}_k n_k \ , \tag{3.10}$$

which is the *Cauchy stress formula.* Equation (3.10) allows us to determine the Cauchy stress vector at some point acting across an arbitrarily inclined plane, if the Cauchy stress vectors acting across the three coordinate surfaces through that point are known.

The stress vectors $\boldsymbol{t}_k$ are, by definition, independent of $\boldsymbol{n}$. From (3.10) it therefore follows that

$$\boldsymbol{t}_{(-\boldsymbol{n})} = -\boldsymbol{t}_{(\boldsymbol{n})} \ . \tag{3.11}$$

Definition (*Cauchy stress tensor*). *The Cauchy stress t_{kl} is the* lth *components of the stress vector $\boldsymbol{t}_k$ acting on the positive side of the* kth *coordinate surface:*

$$\boldsymbol{t}_k = t_{kl}\boldsymbol{i}_l \qquad \text{or} \qquad t_{kl} = \boldsymbol{t}_k \cdot \boldsymbol{i}_l \ . \tag{3.12}$$

The first subscript in t_{kl} indicates the coordinate surface $x_k =$ const. on which the stress vector $\boldsymbol{t}_k$ acts, and the second subscript the direction of the component of $\boldsymbol{t}_k$. For example, t_{23} is the x_3-components of the stress vector $\boldsymbol{t}_2$ acting on the coordinate surface $x_2 =$ const.. Now, if the exterior normal of $x_2 =$ const. points in the positive direction of the x_2-axis, t_{23} points in the positive direction of the x_3-axis. If the exterior normal of $x_2 =$ const. is in the negative direction of the x_2-axis, t_{23} is directed in the negative direction of the x_3-axis. The positive stress components on the faces of a parallelepiped built on coordinate surfaces are shown on Figure 3.3. The components t_{11}, t_{22} and t_{23} are called *normal stresses* and the mixed components t_{12}, t_{13}, etc. are called *shear stresses.* The nine components t_{kl} of the Cauchy stress tensor $\boldsymbol{t}$ may be arranged in a matrix form

$$\boldsymbol{t} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \ . \tag{3.13}$$

Considering (3.12), the Cauchy stress formula (3.10) reads

$$\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{t} \ , \tag{3.14}$$

saying that *the Cauchy stress vector acting on any plane through a point is fully characterized as a linear function of the stress tensor at the point.*

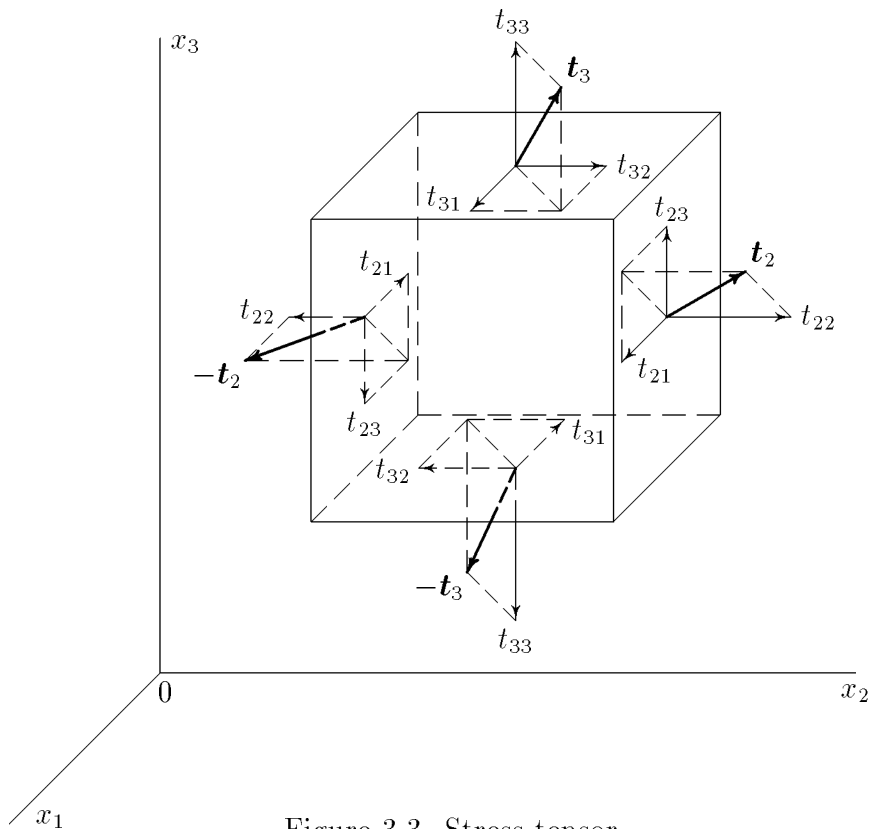

Figure 3.3. Stress tensor.

4. FUNDAMENTAL BALANCE LAWS

4.1 Global balance laws

In continuum mechanics the following five laws are postulated irrespective of material constitution and geometry. They are valid for all bodies subject to thermomechanical effects. The domain of applicability of these laws is restricted by the relativistic speeds (special relativity) and dimensions (general relativity) and microscopic and quantum-mechanical phenomena.

Fundamental Axiom 1 *(Conservation of Mass). The total mass of a body is unchanged with motion.*

This axiom assumes that the mass production and supply is zero. It thus states that the reference (initial) total mass of the body is the same as the total mass of the body at any other time, i.e.,

$$\int_V \varrho_0 \, dV = \int_{v(t)} \varrho \, dv \ , \tag{4.1}$$

where V and $v(t)$ is the reference and the current volume of the body, $\varrho_0(\boldsymbol{X})$ is the mass density of the body in the reference configuration, $\varrho(\boldsymbol{x}, t)$ is the mass density of the body in a current configuration. Using the transformation law $dv = j dV$, we may write this as

$$\int_V \left(\varrho_0 - \varrho\!\!\!\!\varrho\, j\right) \, dV = 0 \ , \tag{4.2}$$

where

$$\varrho\!\!\!\!\varrho(\boldsymbol{X}, t) := \varrho(\boldsymbol{x}(\boldsymbol{X}, t), t) \ . \tag{4.3}$$

Alternatively, we may take the material derivative of (4.1). Thus,

$$\frac{D}{Dt} \int_{v(t)} \varrho \, dv = 0 \ . \tag{4.4}$$

Either expression (4.2) or (4.4) expresses the law of conservation of mass.

Fundamental Axiom 2 *(Balance of Linear Momentum). The time rate of change of the total linear momentum of a body is equal to the resultant force acting on the body.*

Let a body having a current volume $v(t)$ and bounding surface $s(t)$ with exterior unit normal $\boldsymbol{n}$ be subject to surface traction $\boldsymbol{t}_{(\boldsymbol{n})}$ and body force $\boldsymbol{f}$ (body force per unit mass of the body). The resultant force acting on the body is

$$\int_{s(t)} \boldsymbol{t}_{(\boldsymbol{n})} \, da + \int_{v(t)} \varrho \boldsymbol{f} \, dv \ .$$

In addition, let the body be in motion under the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$. The *linear momentum* of the body is defined by the vector

$$\frac{D}{Dt} \int_{v(t)} \varrho \boldsymbol{v} \, dv \ .$$

Thus the balance of linear momentum reads

$$\frac{D}{Dt}\int_{v(t)} \varrho \boldsymbol{v} \, dv = \int_{s(t)} \boldsymbol{t}_{(\boldsymbol{n})} \, da + \int_{v(t)} \varrho \boldsymbol{f} \, dv \ . \tag{4.5}$$

Fundamental Axiom 3 *(Balance of Angular Momentum). The time rate of change of the total angular momentum of a body is equal to the resultant moment of all forces acting on the body.*

Mathematically,

$$\frac{D}{Dt}\int_{v(t)} \varrho \boldsymbol{x} \times \boldsymbol{v} \, dv = \int_{s(t)} \boldsymbol{x} \times \boldsymbol{t}_{(\boldsymbol{n})} \, da + \int_{v(t)} \varrho \boldsymbol{x} \times \boldsymbol{f} \, dv \ , \tag{4.6}$$

where the left-hand side is the time rate of change of the total angular momentum about the origin, which is also frequently called the *moment of momentum*. On the right-hand side the surface integral is the moment of the surface tractions about the origin, and the volume integral is the total moment of body forces about the origin.

Fundamental Axiom 4 *(Conservation of Energy). The time rate of change of the sum of kinetic energy $\mathcal{K}$ and internal energy $\mathcal{E}$ is equal to the sum of the rate of work $\mathcal{W}$ of the surface and body forces and all other energies $\mathcal{U}_\alpha$ that enter and leave body per unit time.*

Mathematically,

$$\frac{D}{Dt}(\mathcal{K} + \mathcal{E}) = \mathcal{W} + \sum_\alpha \mathcal{U}_\alpha \ . \tag{4.7}$$

The total kinetic energy of the body is given by

$$\mathcal{K} = \frac{1}{2}\int_{v(t)} \varrho \boldsymbol{v} \cdot \boldsymbol{v} \, dv \ . \tag{4.8}$$

In continuum mechanics the existence of the internal energy density ε is postulated:

$$\mathcal{E} = \int_{v(t)} \varrho \varepsilon \, dv \ . \tag{4.9}$$

The *mechanical power*, or *rate of work* of the surface traction $\boldsymbol{t}_{(\boldsymbol{n})}$ and body forces $\boldsymbol{f}$ is given by

$$\mathcal{W} = \int_{s(t)} \boldsymbol{t}_{(\boldsymbol{n})} \cdot \boldsymbol{v} \, da + \int_{v(t)} \varrho \boldsymbol{f} \cdot \boldsymbol{v} \, dv \ . \tag{4.10}$$

Other energies $\mathcal{U}_\alpha$ ($\alpha = 1, 2, ..., n$) that enter and leave the body may be of thermal, electromagnetic, chemical, or some other origin. In this text, we consider that the energy transfer in continuum is *thermo-mechanical* and thus only due to work or heat. The heat energy consists of the *heat flux* per unit area $\boldsymbol{q}$ that enters or leaves through the surface of the body and the *heat source* h per unit mass produced by internal sources. Thus we set $\mathcal{U}_\alpha = 0$ except for

$$\mathcal{U}_1 := \mathcal{L} = \int_{s(t)} \boldsymbol{q} \cdot \boldsymbol{n} \; da + \int_{v(t)} \varrho h \; dv \;, \tag{4.11}$$

where the unit normal $\boldsymbol{n}$ is directed outward from the surface of the body. Thus (4.7) reads

$$\frac{D}{Dt} \int_{v(t)} (\varrho \varepsilon + \frac{1}{2} \varrho \boldsymbol{v} \cdot \boldsymbol{v}) \; dv = \int_{s(t)} \left(\boldsymbol{t}_{(\boldsymbol{n})} \cdot \boldsymbol{v} + \boldsymbol{q} \cdot \boldsymbol{n} \right) \; da + \int_{v(t)} (\varrho \boldsymbol{f} \cdot \boldsymbol{v} + \varrho h) \; dv \;, \tag{4.12}$$

which is the statement of the *first law of thermodynamics.*

Fundamental Axiom 5 *(Entropy inequality). The time rate of change of the total entropy H is never less than the sum of the influx of entropy $\boldsymbol{s}$ through the surface of the body and the entropy B supplied by the body forces. This law is postulated to hold for all independent processes.*

Mathematically,

$$\Gamma := \frac{DH}{Dt} - B - \int_{s(t)} \boldsymbol{s} \cdot \boldsymbol{n} \; da \geq 0 \;, \tag{4.13}$$

where Γ so defined is the *total entropy production.* In classical continuum mechanics the entropy density η and entropy source b, per unit mass, are postulated to exist such that

$$H = \int_{v(t)} \varrho \eta \; dv \;, \qquad B = \int_{v(t)} \varrho b \; dv \;.$$

Moreover, we shall be dealing only with *simple thermodynamic processes* for which the entropy flux $\boldsymbol{s}$ and entropy source b are taken as

$$\boldsymbol{s} = \frac{\boldsymbol{q}}{\theta} \;, \qquad b = \frac{h}{\theta} \;. \tag{4.14}$$

where the scalar θ is called the *absolute temperature.* It is subject to

$$\theta > 0 \;, \qquad \inf \theta = 0 \;, \tag{4.15}$$

that is, the temperature is absolute, or it is always positive. Thus, for a simple thermodynamic process the entropy inequality (4.13) reads

$$\frac{D}{Dt} \int_{v(t)} \varrho \eta \; dv - \int_{s(t)} \frac{1}{\theta} \boldsymbol{q} \cdot \boldsymbol{n} \; da - \int_{v(t)} \frac{\varrho h}{\theta} \; dv \geq 0 \;. \tag{4.16}$$

The foregoing five laws are postulated to hold for all bodies irrespective of their geometries and constitutions. To obtain local equations, further restrictions are necessary, which are made in the next section.

4.2 Local balance laws

4.2.1 Continuity equation

We now apply the Reynolds transport theorem (2.23) to the law of conservation of mass (4.4). Considering $\phi = \varrho$ in (2.23), the law of conservation of mass (4.4) reads

$$\int_{v(t)-\sigma(t)} \left(\frac{D\varrho}{Dt} + \varrho \operatorname{div} \boldsymbol{v} \right) dv + \int_{\sigma(t)} [\varrho(\boldsymbol{v} - \boldsymbol{\nu})]_-^+ \cdot \boldsymbol{n}\, da = 0 \ . \tag{4.17}$$

We now assume that the density $\varrho(\boldsymbol{x}, t)$ and $\boldsymbol{v}$ are continuously differentiable functions of the spatial variables x_k and time t in $v(t) - \sigma(t)$ which implies that the integrand of the volume integral in (4.17) is continuous in x_k and t. We also assume that the jump $[\varrho(\boldsymbol{v} - \boldsymbol{\nu})]_-^+$ on the discontinuity surface $\sigma(t)$ is a continuous function of $\boldsymbol{x}$ and t. Moreover, we postulate that *all global balance laws are valid for every part of the body and the discontinuity surface.* Applied to (4.17) this implies that integrands of each of the integral must vanish identically. [2] Thus

$$\begin{aligned} \frac{D\varrho}{Dt} + \varrho \operatorname{div} \boldsymbol{v} &= 0 \qquad \text{in} \ \ v(t) - \sigma(t) \ , & (4.18) \\ [\varrho(\boldsymbol{v} - \boldsymbol{\nu})]_-^+ \cdot \boldsymbol{n} &= 0 \qquad \text{on} \ \ \sigma(t) \ . & (4.19) \end{aligned}$$

These are the equations of local conservation of mass and the jump condition in Eulerian form. Equation (4.18) is often called the *continuity equation.* A different form of (4.18) is obtained by rewriting it as

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{v} \cdot \operatorname{grad} \varrho + \varrho \operatorname{div} \boldsymbol{v} = 0 \ ,$$

or

$$\frac{\partial \varrho}{\partial t} + \operatorname{div} (\varrho \boldsymbol{v}) = 0 \qquad \text{in} \ \ v(t) - \sigma(t) \ . \tag{4.20}$$

The Lagrangian form of continuity equation can be derived from (4.2):

$$\varrho_0 = \varrho j \qquad \text{in} \ \ V \ , \tag{4.21}$$

which is equivalent to

$$\frac{D}{Dt}(\varrho j) = 0 \qquad \text{in} \ \ V \ . \tag{4.22}$$

Let us express the Lagrangian form of the continuity equation (4.21) in the linear theory (the infinitesimal deformation theory). Expressing $\boldsymbol{x}(\boldsymbol{X}$,t) in terms of displacement vector on the right-hand side of (4.3) and expanding the result at the point $\boldsymbol{X}$, we get

$$\varrho(\boldsymbol{X}, t) = \varrho(\boldsymbol{X} + \boldsymbol{u}, t) = \varrho(\boldsymbol{X}, t) + \boldsymbol{u} \cdot \operatorname{grad} \varrho|_{(\boldsymbol{X}, t)} \ , \tag{4.23}$$

[2] For nonlocal continuum theories this postulate is revoked, and only the global balance laws (valid for the entire body) are considered to be valid.

where the second and higher order terms of the Taylor series expansion have been neglected. Due to existence of the mapping (1.1) and eqn.(4.21), the second term on the right-hand side of (4.23) can, within the accuracy of the linear theory, be approximated as

$$\boldsymbol{u} \cdot \operatorname{grad} \varrho|_{(\boldsymbol{X},t)} = \boldsymbol{u} \cdot \operatorname{Grad} \varrho|_{(\boldsymbol{X},t)} = \boldsymbol{u} \cdot \operatorname{Grad} \varrho_0(\boldsymbol{X}) \ . \tag{4.24}$$

Substituting (4.23) and (4.24) into (4.21) and considering that $j \approx 1 + \operatorname{div} \boldsymbol{u}$ in the linear theory, the linearized form of (4.21) reads

$$\varrho(\boldsymbol{X}, t) = \varrho_0(\boldsymbol{X}) - \operatorname{div} \left[\varrho_0(\boldsymbol{X})\boldsymbol{u}(\boldsymbol{X}, t)\right] \ . \tag{4.25}$$

4.2.2 Equation of motion

The equation of global balance of linear momentum (4.5) now reads

$$\frac{D}{Dt}\int_{v(t)-\sigma(t)} \varrho\boldsymbol{v}\ dv = \int_{s(t)-\sigma(t)} \boldsymbol{t}_{(\boldsymbol{n})}\ da + \int_{v(t)-\sigma(t)} \varrho\boldsymbol{f}\ dv \ . \tag{4.26}$$

By substituting for the Cauchy stress vector $\boldsymbol{t}_{(\boldsymbol{n})}$ from (3.14) and using the generalized Gauss's theorem (2.20), we obtain

$$\frac{D}{Dt}\int_{v(t)-\sigma(t)} \varrho\boldsymbol{v}\ dv = \int_{v(t)-\sigma(t)} (\operatorname{div}\boldsymbol{t} + \varrho\boldsymbol{f})\ dv + \int_{\sigma(t)} \boldsymbol{n}\cdot[\boldsymbol{t}]_-^+\ da \ . \tag{4.27}$$

Using Reynolds's transport teorem (2.23) with $\phi = \varrho\boldsymbol{v}$, we obtain

$$\int_{v(t)-\sigma(t)} \left(\frac{D(\varrho\boldsymbol{v})}{Dt} + \varrho\boldsymbol{v}\operatorname{div}\boldsymbol{v} - \operatorname{div}\boldsymbol{t} - \varrho\boldsymbol{f}\right)\ dv + \int_{\sigma(t)} \left[\varrho\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{\nu}) - \boldsymbol{t}^T\right]_-^+ \cdot \boldsymbol{n}\ da = 0 \ . \tag{4.28}$$

This is postulated to be valid for all parts of the body. Thus the integrands vanish separately. Upon using (4.18), this is simplified to

$$\operatorname{div}\boldsymbol{t} + \varrho\boldsymbol{f} = \varrho\frac{D\boldsymbol{v}}{Dt} \qquad \text{in } \ v(t)-\sigma(t)\ , \tag{4.29}$$

$$\left[\varrho\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{\nu}) - \boldsymbol{t}^T\right]_-^+ \cdot \boldsymbol{n} = 0 \qquad \text{on } \ \sigma(t)\ . \tag{4.30}$$

Equation (4.29) is known as *Cauchy's equations of motion* in Eulerian form expressing the local balance of linear momentum, and (4.30) is the associated jump condition on the singular surface σ.

Alternatively, in view of the definition of $\operatorname{div}\boldsymbol{t}$, $\operatorname{div}\boldsymbol{t} = t_{kl,k}\boldsymbol{i}_l$, and of the differentiation of (3.12) with respect to x_k, we observe that

$$\operatorname{div}\boldsymbol{t} = \boldsymbol{t}_{k,k} \ . \tag{4.31}$$

Thus the Cauchy equation of motion (4.29) can be expressed in terms of the stress vectors $\boldsymbol{t}_k$ as

$$\boldsymbol{t}_{k,k} + \varrho \boldsymbol{f} = \varrho \frac{D\boldsymbol{v}}{Dt} \qquad \text{in} \quad v(t) - \sigma(t) \; . \tag{4.32}$$

4.2.3 Symmetry of the Cauchy stress tensor

The angular momentum of the surface tractions about the origin occurring in the global law of balance of the angular momentum (4.6), can be rewritten by using the Cauchy stress formula (3.14), the tensor identity

$$\boldsymbol{v} \times (\boldsymbol{w} \cdot \boldsymbol{A}) = -\boldsymbol{w} \cdot (\boldsymbol{A} \times \boldsymbol{v}) \; , \tag{4.33}$$

where $\boldsymbol{v}$, $\boldsymbol{w}$ are vectors, $\boldsymbol{A}$ is a second-order tensor, and the generalized Gauss theorem (2.20), to the form

$$\int_{s(t)-\sigma(t)} \boldsymbol{x} \times \boldsymbol{t}_{(\boldsymbol{n})} \, da = -\int_{v(t)-\sigma(t)} \operatorname{div}(\boldsymbol{t} \times \boldsymbol{x}) \, dv - \int_{\sigma(t)} \boldsymbol{n} \cdot [\boldsymbol{t} \times \boldsymbol{x}]_-^+ \, da \; . \tag{4.34}$$

By making use of the two tensor identities,

$$\operatorname{div}(\boldsymbol{t} \times \boldsymbol{v}) = \operatorname{div} \boldsymbol{t} \times \boldsymbol{v} + \boldsymbol{t}^T \overset{\bullet}{\times} \operatorname{grad} \boldsymbol{v} \; , \tag{4.35}$$

$$\operatorname{grad} \boldsymbol{x} = \boldsymbol{I} \; , \tag{4.36}$$

where the superscript T at tensor $\boldsymbol{t}$ stands for transposition, $\overset{\bullet}{\times}$ denotes the dot-cross product of the 2nd order tensors, and $\boldsymbol{I}$ is the second-order identity tensor, we further have

$$\int_{v(t)-\sigma(t)} \operatorname{div}(\boldsymbol{t} \times \boldsymbol{x}) \, dv = \int_{v(t)-\sigma(t)} \left(\operatorname{div} \boldsymbol{t} \times \boldsymbol{x} + \boldsymbol{t}^T \overset{\bullet}{\times} \boldsymbol{I} \right) \, dv \; .$$

Upon carrying this and (4.34) into the equation of balance of the angular momentum (4.6) and using the Reynolds transport theorem (2.23) with $\phi = \boldsymbol{x} \times \varrho \boldsymbol{v}$, we obtain

$$\int_{v(t)-\sigma(t)} \left(\frac{D(\varrho \boldsymbol{x} \times \boldsymbol{v})}{Dt} + (\varrho \boldsymbol{x} \times \boldsymbol{v}) \operatorname{div} \boldsymbol{v} + \operatorname{div} \boldsymbol{t} \times \boldsymbol{x} + \boldsymbol{t}^T \overset{\bullet}{\times} \boldsymbol{I} - \varrho \boldsymbol{x} \times \boldsymbol{f} \right) \, dv +$$

$$+ \int_{\sigma(t)} [\boldsymbol{x} \times \varrho \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{\nu})]_-^+ \cdot \boldsymbol{n} \, da + \int_{\sigma(t)} \boldsymbol{n} \cdot [\boldsymbol{t} \times \boldsymbol{x}]_-^+ \, da = 0 \; ,$$

which can be arranged to the form

$$\int_{v(t)-\sigma(t)} \left[(\boldsymbol{x} \times \boldsymbol{v}) \left(\frac{D\varrho}{Dt} + \varrho \operatorname{div} \boldsymbol{v} \right) + \varrho \frac{D\boldsymbol{x}}{Dt} \times \boldsymbol{v} + \boldsymbol{x} \times \left(\varrho \frac{D\boldsymbol{v}}{Dt} - \operatorname{div} \boldsymbol{t} - \varrho \boldsymbol{f} \right) + \boldsymbol{t}^T \overset{\bullet}{\times} \boldsymbol{I} \right] \, dv +$$

$$+ \int_{\sigma(t)} \boldsymbol{x} \times \left[\varrho \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{\nu}) - \boldsymbol{t}^T \right]_-^+ \cdot \boldsymbol{n} \, da = 0 \; . \tag{4.37}$$

Considering

$$\frac{D\boldsymbol{x}}{Dt} \times \boldsymbol{v} = \boldsymbol{v} \times \boldsymbol{v} = 0$$

and using the local laws of conservation of mass (4.18), the balance of linear momentum (4.29), and the associated jump condition (4.30), we get

$$\int_{v(t)-\sigma(t)} \boldsymbol{t}^T \overset{\bullet}{\times} \boldsymbol{I} \; dv = 0 \; . \tag{4.38}$$

Again, postulating that this to be valid for all parts of $v(t) - \sigma(t)$, the integrand must vanish, so that,

$$\boldsymbol{t}^T \overset{\bullet}{\times} \boldsymbol{I} = 0 \qquad \text{or} \qquad \boldsymbol{t}^T = \boldsymbol{t} \qquad \text{in} \;\; v(t) - \sigma(t) \; . \tag{4.39}$$

Thus the necessary and sufficient condition for the satisfaction of the local balance of angular momentum is the symmetry of the Cauchy stress tensor $\boldsymbol{t}$. We have seen that the associated jump condition for the angular momentum is satisfied identically.

Note that in formulating the angular principle by (4.6) we have assumed that no body nor surface couples act on the body. If any such concentrated moments do act, the material is said to be a *polar* material and the symmetry property of $\boldsymbol{t}$ no longer holds. But this is a rather specialized situation and we shall not consider it here.

4.2.4 Energy equation

The same program can be carried out for the equation of energy balance (4.12). The surface integral occurring on the right-hand side of the equation of energy balance can be rewritten by using the Cauchy stress formula (3.14) and then can be converted to the volume integral by the Gauss theorem for a vector to the form

$$\int_{s(t)-\sigma(t)} \left(\boldsymbol{t}_{(\boldsymbol{n})} \cdot \boldsymbol{v} + \boldsymbol{q} \cdot \boldsymbol{n}\right) \; da = \int_{v(t)-\sigma(t)} \left(\operatorname{div}(\boldsymbol{t} \cdot \boldsymbol{v}) + \operatorname{div} \boldsymbol{q}\right) \; dv + \int_{\sigma(t)} [\boldsymbol{t} \cdot \boldsymbol{v} + \boldsymbol{q}]^+_- \cdot \boldsymbol{n} \; da \; . \tag{4.40}$$

The divergence of vector $\boldsymbol{t} \cdot \boldsymbol{v}$ will be arranged by making use of the identity:

$$\operatorname{div}(\boldsymbol{t} \cdot \boldsymbol{v}) = \operatorname{div} \boldsymbol{t} \cdot \boldsymbol{v} + \boldsymbol{t}^T : \operatorname{grad} \boldsymbol{v} \; , \tag{4.41}$$

where $:$ stands for the double-dot product of tensors. The left-hand side of the equation of energy balance (4.12) can be arranged by Reynolds's transport theorem (2.23) as

$$\frac{D}{Dt} \int_{v(t)-\sigma(t)} (\varrho\varepsilon + \frac{1}{2}\varrho \boldsymbol{v} \cdot \boldsymbol{v}) \; dv = \int_{v(t)-\sigma(t)} \left[\frac{D}{Dt}(\varrho\varepsilon + \frac{1}{2}\varrho\boldsymbol{v} \cdot \boldsymbol{v}) + (\varrho\varepsilon + \frac{1}{2}\varrho\boldsymbol{v} \cdot \boldsymbol{v})\operatorname{div} \boldsymbol{v}\right] \; dv +$$

$$+ \int_{\sigma(t)} \left[(\varrho\varepsilon + \frac{1}{2}\varrho\boldsymbol{v} \cdot \boldsymbol{v})(\boldsymbol{v} - \boldsymbol{\nu})\right]^+_- \cdot \boldsymbol{n} \; da =$$

$$= \int_{v(t)-\sigma(t)} \left[(\varepsilon + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}) \left(\frac{D\varrho}{Dt} + \varrho \operatorname{div} \boldsymbol{v} \right) + \varrho \frac{D\varepsilon}{Dt} + \varrho \frac{D\boldsymbol{v}}{Dt} \cdot \boldsymbol{v} \right] \, dv +$$

$$+ \int_{\sigma(t)} \left[(\varrho\varepsilon + \frac{1}{2}\varrho\boldsymbol{v}\cdot\boldsymbol{v})(\boldsymbol{v} - \boldsymbol{\nu}) \right]_-^+ \cdot \boldsymbol{n} \, da \ ,$$

which, by the law of mass conservation (4.18), reduces to

$$\frac{D}{Dt} \int_{v(t)-\sigma(t)} (\varrho\varepsilon + \frac{1}{2}\varrho\boldsymbol{v}\cdot\boldsymbol{v}) \, dv = \int_{v(t)-\sigma(t)} \left(\varrho \frac{D\varepsilon}{Dt} + \varrho \frac{D\boldsymbol{v}}{Dt} \cdot \boldsymbol{v} \right) \, dv +$$

$$+ \int_{\sigma(t)} \left[(\varrho\varepsilon + \frac{1}{2}\varrho\boldsymbol{v}\cdot\boldsymbol{v})(\boldsymbol{v} - \boldsymbol{\nu}) \right]_-^+ \cdot \boldsymbol{n} \, da \ . \tag{4.42}$$

In view of (4.40)–(4.42), the equation of motion (4.29), the symmetry of the Cauchy stress tensor, and upon setting the integrand of the result equal to zero, we obtain

$$\varrho \frac{D\varepsilon}{Dt} = \boldsymbol{t} : \operatorname{grad} \boldsymbol{v} + \operatorname{div} \boldsymbol{q} + \varrho \mathrm{h} \qquad \text{in} \quad v(t) - \sigma(t) \ , \tag{4.43}$$

$$\left[(\varrho\varepsilon + \frac{1}{2}\varrho\boldsymbol{v}\cdot\boldsymbol{v})(\boldsymbol{v} - \boldsymbol{\nu}) - \boldsymbol{t}\cdot\boldsymbol{v} - \boldsymbol{q} \right]_-^+ \cdot \boldsymbol{n} = 0 \qquad \text{on} \quad \sigma(t) \ . \tag{4.44}$$

In view of the symmetry of the Cauchy stress tensor $\boldsymbol{t}$, we further have

$$\varrho \frac{D\varepsilon}{Dt} = \boldsymbol{t} : \boldsymbol{d} + \operatorname{div} \boldsymbol{q} + \varrho h \qquad \text{in} \quad v(t) - \sigma(t) \ , \tag{4.45}$$

where $\boldsymbol{d}$ is the strain-rate tensor introduced by (2.13). Equation (4.45) is the energy equation for a *thermomechanical continuum* and (4.44) is the associated jump condition on the singular surface σ.

4.2.5 Entropy inequality

The same program can be applied to the global law of entropy to carry out it to the local form. Using again the Reynolds transport theorem, the Cauchy stress formula and the Gauss theorem, and assuming that the global law of entropy (4.13) is valid for any part of the body, we get the local production of entropy

$$\varrho \frac{D\eta}{Dt} - \operatorname{div} \boldsymbol{s} - \varrho b \ \geq \ 0 \qquad \text{in} \quad v(t) - \sigma(t) \ , \tag{4.46}$$

$$[\varrho\eta(\boldsymbol{v} - \boldsymbol{\nu}) - \boldsymbol{s}]_-^+ \cdot \boldsymbol{n} \ \geq \ 0 \qquad \text{on} \quad \sigma(t) \ . \tag{4.47}$$

For a simple thermodynamical process, the entropy influx $\boldsymbol{s}$ and entropy source b can be taken according to (4.14), and heat source h can be eliminated between (4.45) and (4.46). The entropy inequality then takes the form

$$\varrho \left(\frac{D\eta}{Dt} - \frac{1}{\theta} \frac{D\varepsilon}{Dt} \right) + \frac{1}{\theta} \boldsymbol{t} : \boldsymbol{d} + \frac{1}{\theta^2} \operatorname{grad} \theta \cdot \boldsymbol{q} \geq 0 \ , \tag{4.48}$$

which is known as the *Clausius-Duhem inequality.*

4.2.6 Résumé of local balance laws

For a quick reference, we collect all local balance laws.

(i) *Conservation of mass*

$$\begin{aligned} \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) &= 0 && \text{in } \ v(t) - \sigma(t)\ , && (4.49)\\ [\varrho(\boldsymbol{v} - \boldsymbol{\nu})]_-^+ \cdot \boldsymbol{n} &= 0 && \text{on } \ \sigma(t)\ . && (4.50) \end{aligned}$$

(ii) *Balance of linear momentum*

$$\begin{aligned} \operatorname{div} \boldsymbol{t} + \varrho \boldsymbol{f} &= \varrho \frac{D\boldsymbol{v}}{Dt} && \text{in } \ v(t) - \sigma(t)\ , && (4.51)\\ [\varrho \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{\nu}) - \boldsymbol{t}]_-^+ \cdot \boldsymbol{n} &= 0 && \text{on } \ \sigma(t)\ . && (4.52) \end{aligned}$$

(iii) *Balance of angular momentum*

$$\boldsymbol{t} = \boldsymbol{t}^T \qquad \text{in } \ v(t) - \sigma(t)\ . \qquad (4.53)$$

(iv) *Conservation of energy*

$$\varrho \frac{D\varepsilon}{Dt} = \boldsymbol{t} : \boldsymbol{d} + \operatorname{div} \boldsymbol{q} + \varrho h \qquad \text{in } \ v(t) - \sigma(t)\ , \qquad (4.54)$$

$$\left[(\varrho\varepsilon + \frac{1}{2}\varrho \boldsymbol{v} \cdot \boldsymbol{v})(\boldsymbol{v} - \boldsymbol{\nu}) - \boldsymbol{t} \cdot \boldsymbol{v} - \boldsymbol{q}\right]_-^+ \cdot \boldsymbol{n} = 0 \qquad \text{on } \ \sigma(t)\ . \qquad (4.55)$$

(v) *Entropy inequality*

$$\begin{aligned} \varrho \frac{D\eta}{Dt} - \operatorname{div} \boldsymbol{s} - \varrho b &\geq 0 && \text{in } \ v(t) - \sigma(t)\ , && (4.56)\\ [\varrho\eta(\boldsymbol{v} - \boldsymbol{\nu}) - \boldsymbol{s}]_-^+ \cdot \boldsymbol{n} &\geq 0 && \text{on } \ \sigma(t)\ . && (4.57) \end{aligned}$$

4.3 Jump conditions in special cases

If there is a moving discontinuity surface $\sigma(t)$ sweeping the body with a velocity $\boldsymbol{\nu}$ in the direction of the unit normal $\boldsymbol{n}$ of $\sigma(t)$, then the jump conditions (4.50), (4.52) (4.55) and (4.57) must be satisfied on the surface $\sigma(t)$. Some of these jump conditions will now be applied to two special cases:

(i) **The discontinuity surface is a material surface.** In this case, $\boldsymbol{\nu} = \boldsymbol{v}$, (4.50) is satisfied identically, (4.52) and (4.55) reduce to

$$\begin{aligned} [\boldsymbol{t}]_-^+ \cdot \boldsymbol{n} &= \boldsymbol{0}\ , && (4.58)\\ [\boldsymbol{t} \cdot \boldsymbol{v} + \boldsymbol{q}]_-^+ \cdot \boldsymbol{n} &= 0\ . && (4.59) \end{aligned}$$

Hence, on a material interface between two media the surface traction $\boldsymbol{t} \cdot \boldsymbol{n}$ is continuous, and the jump on the energy of tractions across this interface is balanced with that of the normal component of the heat vector.

A *chemical interface* of two media is usually taken as the material interface with $\boldsymbol{\nu} = \boldsymbol{v}$ across which the material velocity is continuous $[\boldsymbol{v}]_-^+ = 0$. Condition (4.59) is then further reduced by (4.58) to

$$[\boldsymbol{q}]_-^+ \cdot \boldsymbol{n} = 0 \ , \tag{4.60}$$

which states the continuity in the normal component of the heat vector across $\sigma(t)$.

A *free-slip interface* of two materials is the material interface across which the motion from one of its side runs without friction. This means that the shear stresses of the Cauchy stress tensor $\boldsymbol{t}$ are equal to zero from one of interface side (e.g., with superscript '$^-$'),

$$\boldsymbol{n}^- \cdot \boldsymbol{t}^- \cdot \left(\boldsymbol{I} - \boldsymbol{n}^-\boldsymbol{n}^-\right) = \boldsymbol{0} \qquad \text{or} \qquad \boldsymbol{n}^- \cdot \boldsymbol{t}^- = \left(\boldsymbol{n}^- \cdot \boldsymbol{t}^- \cdot \boldsymbol{n}^-\right)\boldsymbol{n}^- \ . \tag{4.61}$$

Carrying this into (4.58) and considering that $\boldsymbol{n} = \boldsymbol{n}^- = -\boldsymbol{n}^+$ across $\sigma(t)$, we get

$$[\boldsymbol{n} \cdot \boldsymbol{t} \cdot \boldsymbol{n}]_-^+ = 0 \ , \tag{4.62}$$

which states the continuity of the normal stress of tensor $\boldsymbol{t}$ across discontinuity surface $\sigma(t)$.

(ii) **The discontinuity surface coincides with the surface of the body.** In this case $\varrho^+ = 0$, $\boldsymbol{v}^- = \boldsymbol{\nu}$. Again (4.50) gives an identity and the others reduce to

$$[\boldsymbol{t}]_-^+ \cdot \boldsymbol{n} \ = \ \boldsymbol{0} \ , \tag{4.63}$$

$$[\boldsymbol{t} \cdot \boldsymbol{v} + \boldsymbol{q}]_-^+ \cdot \boldsymbol{n} \ = \ 0 \ , \tag{4.64}$$

where $\boldsymbol{t}^+ \cdot \boldsymbol{n}$ is interpreted as the external surface load and $\boldsymbol{t}^+ \cdot \boldsymbol{v}^+$ as the energy of this load. If the external surface load is equal to zero, $\boldsymbol{t}^+ = 0$, then $\boldsymbol{t}^- \cdot \boldsymbol{n} = \boldsymbol{0}$, and the first term on the left of (4.64) is equal to zero. Hence, we obtain the boundary condition

$$[\boldsymbol{q}]_-^+ \cdot \boldsymbol{n} = 0 \tag{4.65}$$

involving the heat alone.

4.4 Equation of motion in the reference frame

Any surface forces existing in a continuum are, in general, associated with deformation. When introducing the concept of traction, it is thus natural to reckon the forces acting in the deformed state across some surface per unit area of this surface. This view has led to the definition of the Cauchy stress tensor, which is given in the Eulerian formulation (Sec.3.2). If an undeformed state can be distinguished in the continuum, the use of the Lagrangian formulation with this state serving as the reference state may be more convenient to employ.

The Lagrangian formulation of the balance of linear momentum is based upon the Piola-Kirchhoff stress vector and tensor, which we now introduce. Let $\boldsymbol{T}_K$ be the stress vector at a spatial point $\boldsymbol{x}$ at time t occupied by a material point $\boldsymbol{X}$ in the undeformed area dA_K:

$$\boldsymbol{t}_{(\boldsymbol{n})}da = \boldsymbol{t}_k(\boldsymbol{x},t)da_k = \boldsymbol{T}_K(\boldsymbol{X},t)dA_K \ . \tag{4.66}$$

Using (1.101) we obtain from this

$$\boldsymbol{t}_k = j^{-1}x_{k,K}\boldsymbol{T}_K \ , \qquad \boldsymbol{T}_K = jX_{K,k}\boldsymbol{t}_k \ . \tag{4.67}$$

If we now substitute the first of these into (4.32), we obtain

$$\left(j^{-1}x_{k,K}\right)_{,k}\boldsymbol{T}_k + j^{-1}x_{k,K}\boldsymbol{T}_{K,k} + \varrho\boldsymbol{f} = \varrho\,\frac{D\boldsymbol{v}}{Dt} \ .$$

In view of $(1.24)_2$, the first term on the left-hand side is equal to zero. Furthermore, using (4.3), (4.66) , and introducing

$$\boldsymbol{F}(\boldsymbol{X},t) := \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{X},t),t) \tag{4.68}$$

we get

$$\boldsymbol{T}_{K,K} + \varrho_0\boldsymbol{F} = \varrho_0\frac{D\boldsymbol{V}}{Dt} \qquad \text{in } \ V. \tag{4.69}$$

This is the Cauchy's equation of motion in the reference (Lagrangian) form. Note that it *formally* agrees with the Eulerian form of the equation of motion (4.32).

For component representation we introduce the *first and second Piola-Kirchhoff stress tensors* T_{kl} and T_{KL} by

$$\boldsymbol{T}_K = T_{Kl}\boldsymbol{i}_l = T_{KL}x_{l,L}\boldsymbol{i}_l \ , \tag{4.70}$$

so that by (4.67) we have

$$T_{Kl} = jX_{K,k}t_{kl} \qquad \text{or} \qquad \boldsymbol{T}^{(1)} = j\boldsymbol{F}^{-1}\cdot\boldsymbol{t} \ , \tag{4.71}$$

$$T_{KL} = T_{Kl}X_{L,l} = jX_{K,k}X_{L,l}t_{kl} \qquad \text{or} \qquad \boldsymbol{T}^{(2)} = \boldsymbol{T}^{(1)}\cdot\left(\boldsymbol{F}^{-1}\right)^T = j\boldsymbol{F}^{-1}\cdot\boldsymbol{t}\cdot\left(\boldsymbol{F}^{-1}\right)^T \ , \tag{4.72}$$

$$t_{kl} = \frac{1}{j}x_{k,K}T_{Kl} = \frac{1}{j}x_{k,K}x_{l,L}T_{KL} \qquad \text{or} \qquad \boldsymbol{t} = \frac{1}{j}\boldsymbol{F}\cdot\boldsymbol{T}^{(1)} = \frac{1}{j}\boldsymbol{F}\cdot\boldsymbol{T}^{(2)}\cdot\boldsymbol{F}^T \ , \tag{4.73}$$

where $\boldsymbol{T}^{(1)}$ and $\boldsymbol{T}^{(2)}$ is the symbolic notation for the first and second Piola-Kirchhoff tensor, respectively. From (4.70) it is now clear that T_{Kl} is the stress at $\boldsymbol{x}$ measured per unit undeformed area at $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x},t)$.

Using (4.70) in (4.69) we obtain two different forms of the equations of motion:

$$T_{Kl,K} + \varrho_0 F_l = \varrho_0\frac{DV_l}{Dt} \qquad \text{in } \ V, \tag{4.74}$$

$$(T_{KL}x_{l,L})_{,K} + \varrho_0 F_l = \varrho_0\frac{DV_l}{Dt} \qquad \text{in } \ V. \tag{4.75}$$

Cauchy's second law of motion follows from $t_{kl} = t_{lk}$ upon using (4.73). Hence in two different forms we have

$$T_{Kl}x_{k,K} = T_{Kk}x_{l,K} \ , \tag{4.76}$$

$$T_{KL} = T_{LK} \ . \tag{4.77}$$

The foregoing expressions may be used as a source for approximate theories in which displacement gradient $\boldsymbol{H}$ is much small as compared to unity that the linearization is justified. To this end, we carry the linearized forms (1.60) and (1.61) into (4.71) and (4.72), and we obtain

$$\boldsymbol{T}^{(1)} = (1 + \operatorname{tr}\boldsymbol{H})\,\boldsymbol{t} - \boldsymbol{H}^T \cdot \boldsymbol{t} + O\left(\delta^2\right) \ , \tag{4.78}$$

$$\boldsymbol{T}^{(2)} = (1 + \operatorname{tr}\boldsymbol{H})\,\boldsymbol{t} - \boldsymbol{H}^T \cdot \boldsymbol{t} - \boldsymbol{t} \cdot \boldsymbol{H} + O\left(\delta^2\right) \ . \tag{4.79}$$

The last equation demonstrates that the symmetry of tensor $\boldsymbol{T}^{(2)}$ has not been violated by linearization process.

Supposing, in addition, that stresses are small as compared to unity (the *infinitesimal deformation and stress theory*), then

$$\boldsymbol{T}^{(1)} \cong \boldsymbol{T}^{(2)} \cong \boldsymbol{t} \ , \tag{4.80}$$

showing that, in the infinitesimal deformation and stress theory, there is no difference between the Cauchy and the Piola-Kirchhoff stresses.

5. CONSTITUTIVE EQUATIONS

5.1 The need for constitutive equations

Basic principles of continuum mechanics, namely, conservation of mass, balance of momenta, conservation of energy, and entropy inequality, discussed in Chapter 4, lead to the fundamental equations:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0 \ , \tag{5.1}$$

$$\operatorname{div} \boldsymbol{t} + \varrho \boldsymbol{f} = \varrho \frac{D\boldsymbol{v}}{Dt} \ , \tag{5.2}$$

$$\boldsymbol{t} = \boldsymbol{t}^T \ , \tag{5.3}$$

$$\varrho \frac{D\varepsilon}{Dt} = \boldsymbol{t} : \boldsymbol{d} + \operatorname{div} \boldsymbol{q} + \varrho h \ , \tag{5.4}$$

$$\varrho \left(\frac{D\eta}{Dt} - \frac{1}{\theta} \frac{D\varepsilon}{Dt} \right) + \frac{1}{\theta} \boldsymbol{t} : \boldsymbol{d} + \frac{1}{\theta^2} \operatorname{grad} \theta \cdot \boldsymbol{q} \geq 0 \ . \tag{5.5}$$

In total, they constitute eight independent equations (one for mass, three for linear momentum, three for angular momentum and one for energy) and one inequality. The number of unknowns ϱ, v_k, t_{kl}, q_k, ε, and η, are eighteen provided that body forces f_k and distribution of heat sources h are given. Ten additional equations must be given in order the system to be determinate except for some trivial situations, for example, rigid body motions in the absence of heat conduction.

In the derivation of the equations (5.1) to (5.5) no differentiation has been made between various types materials. It is therefore not surprising that the foregoing equations are not sufficient to explain fully the motions of materials having various type of physical properties. The character of the material is brought into the formulation through the so-called *constitutive equations*, which specify the mechanical and thermal properties of particular materials based upon their internal constitution. Mathematically, the usefulness of these constitutive equations is to describe the relationships among the kinematic, mechanical, and thermal field variables and to permit the formulations of well-posed problems of continuum mechanics. Physically, the constitutive equations define various idealized materials which serve as models for the behavior of real materials. However, it is not possible to write one equation capable of representing a given material over its entire range of application, since many materials behave quite differently under changing levels of loading, such as elastic-plastic response due to increasing stress. Thus, in this sense it is perhaps better to think of constitutive equations as representative of a particular *behavior* rather than of a particular *material*.

In this text we deal with the constitutive equations of *thermomechanical materials*. The study of the chemical changes and electromagnetic effects are excluded. A large class of materials does not undergo chemical transition or produce appreciable electromagnetic effects when deformed. However, the deformation and motion generally produce heat.

Conversely, materials subjected to thermal changes deform and flow. The effect of thermal changes on the material behavior depends on the range and severity of such changes.

The thermomechanical constitutive equations are relations between a set of thermomechanical variables. They may be expressed as

$$\begin{aligned} \boldsymbol{t}(\boldsymbol{X},t) &= \mathcal{F}(\boldsymbol{x}(\boldsymbol{X}',\tau),\theta(\boldsymbol{X}',\tau),\boldsymbol{X},t)\ , \\ \boldsymbol{q}(\boldsymbol{X},t) &= \mathcal{Q}(\boldsymbol{x}(\boldsymbol{X}',\tau),\theta(\boldsymbol{X}',\tau),\boldsymbol{X},t)\ , \\ \varepsilon(\boldsymbol{X},t) &= \mathcal{E}(\boldsymbol{x}(\boldsymbol{X}',\tau),\theta(\boldsymbol{X}',\tau),\boldsymbol{X},t)\ , \\ \eta(\boldsymbol{X},t) &= \mathcal{N}(\boldsymbol{x}(\boldsymbol{X}',\tau),\theta(\boldsymbol{X}',\tau),\boldsymbol{X},t)\ . \end{aligned} \tag{5.6}$$

Note that all response functions $\mathcal{F}$, $\mathcal{Q}$, $\mathcal{E}$ and $\mathcal{N}$ are assumed to depend on the same set of dependent variables $\boldsymbol{x}(\boldsymbol{X}',\tau)$, $\theta(\boldsymbol{X}',\tau)$, $\boldsymbol{X}$ and t. This is known as the **axiom of equipresence**.

5.2 A general mechanical constitutive equation

A *constitutive equation* is a relation between a set of termomechanical variables. In purely mechanical theory of a one-component system, which is taken as the first example, a constitutive equation relates stress and strain tensors. Three fundamental axioms are assumed to be valid for any constitutive theory of purely mechanical phenomena:

- **Axiom of determinism:** The present state of the stress at a material point $\boldsymbol{X}$ of the body $\mathcal{B}$ at time t is uniquely determined by the past history of the motion of all material points of the body $\mathcal{B}$.

- **Axiom of local action:** The motion at distant material points from $\boldsymbol{X}$ does not affect appreciably the stress at $\boldsymbol{X}$.

- **Axiom of frame indifference** or **axiom of objectivity**: A constitutive equation must be form-invariant under rigid motions of the spatial frame of reference.

According to the first axiom, the mechanical constitutive equation may be written as

$$\boldsymbol{t}(\boldsymbol{X},t) = \underset{\substack{\boldsymbol{X}'\in\mathcal{B} \\ \tau\le t}}{\mathcal{F}}\ [\boldsymbol{x}(\boldsymbol{X}',\tau),\boldsymbol{X},t]\ . \tag{5.7}$$

where the *response functional* $\mathcal{F}$ is a tensor-valued function. This relation states, as a starting assumption, an interaction between the past histories of the internal forces and the motion of all materials points of the body $\mathcal{B}$. The internal forces are represented by the Cauchy stress tensor $\boldsymbol{t}$; this may be justified by the local form of the balance of linear momentum. The axiom of determinism is a principle of exclusions. It excludes the dependence of the material behavior at $\boldsymbol{X}$ on any point outside the body and any future events. Consequently,

$$\boldsymbol{X}'\in\mathcal{B}\ , \qquad \tau\le t\ , \tag{5.8}$$

where τ are all past times and t is the present time.

The general functional $\mathcal{F}[\cdot]$ is subject to the other two axioms. The axiom objectivity states that the material properties cannot depend on the motion of the observer. If two spatial frames $\boldsymbol{x}^*$ and $\boldsymbol{x}$ can be made to coincide with a rigid motion, then they must be related to each other as

$$\boldsymbol{x}^*(\boldsymbol{X}, t^*) = \boldsymbol{Q}(t) \cdot \boldsymbol{x}(\boldsymbol{X}, t) + \boldsymbol{b}(t) \ , \qquad t^* = t - a \ , \tag{5.9}$$

where $\boldsymbol{Q}(t)$ is a time-dependent orthogonal tensor,

$$\boldsymbol{Q}(t) \cdot \boldsymbol{Q}^T(t) = \boldsymbol{Q}^T(t) \cdot \boldsymbol{Q}(t) = \boldsymbol{I} \ , \qquad \det \boldsymbol{Q} = 1 \ , \tag{5.10}$$

$\boldsymbol{b}(t)$ is a time-dependent vector and a is a constant. Equations (5.9) express general rigid motion of the spatial frame of reference and shift of the origin of time. In fact, $\boldsymbol{b}(t)$ corresponds to the translation, $\boldsymbol{Q}(t)$ to the rotation of the spatial frame of reference and a is a constant shift of the origin of time.

Differentiating eqn.(5.9) with respect to $\boldsymbol{X}$ we can show that the Cauchy stress tensor transforms under rigid motion of frame according to the relation

$$\boldsymbol{t}^*(\boldsymbol{X}, t^*) = \boldsymbol{Q}(t) \cdot \boldsymbol{t}(\boldsymbol{X}, t) \cdot \boldsymbol{Q}^T(t). \tag{5.11}$$

The transformation rule for the deformation gradient is readily to show that

$$\boldsymbol{F}^*(X, t) = \boldsymbol{Q}(t) \cdot \boldsymbol{F}(X, t) \ , \tag{5.12}$$

that is, this two-point tensor transforms like a vector under a rigid motion of frame at time t. In view of the transformation of physical quantities in the context of a rigid motion of frame, we call scalar-, vector and tensor-valued physical quantities ϕ, $\boldsymbol{v}$ and $\boldsymbol{A}$ *objective* if they transform under a rigid motion of frame according to

$$\begin{aligned} \phi^* &= \phi \ , \\ \boldsymbol{v}^* &= \boldsymbol{Q} \cdot \boldsymbol{v} \ , \\ \boldsymbol{A}^* &= \boldsymbol{Q} \cdot \boldsymbol{A} \cdot \boldsymbol{Q}^T \ . \end{aligned} \tag{5.13}$$

In the sense of this definition, the mass of a material body is an objective scalar, the force is an objective vector and the Cauchy stress tensor is an objective tensor. Quantities which are not objective are, among others, the position vector, the velocity vector and the deformation gradient. The property of a particular quantity to be objective or not is either a priori postulated or derived from other definitions.

According to the axiom of objectivity, the form of functional $\mathcal{F}$ should be the same in any two objectively equivalent motions, that is

$$\boldsymbol{t}^*(\boldsymbol{X}, t^*) = \underset{\substack{\boldsymbol{X}' \in \mathcal{B} \\ \tau^* \le t^*}}{\mathcal{F}} \left[\boldsymbol{x}^*(\boldsymbol{X}', \tau^*), \boldsymbol{X}, t^*\right] \ . \tag{5.14}$$

Note that no star is attached to the functional $\mathcal{F}$: The axiom of frame indifference states that an objective change of frame has no influence on the functional $\mathcal{F}$ which maps the deformation history into the present state of stress. Substituting eqns.(5.7) and (5.11) into (5.14) we can conclude that the functional $\mathcal{F}$ must satisfy a restrictive condition of the form

$$\boldsymbol{Q}(t) \cdot \underset{\substack{\boldsymbol{X}' \in \mathcal{B} \\ \tau \leq t}}{\mathcal{F}} [\boldsymbol{x}(\boldsymbol{X}', \tau), \boldsymbol{X}, t] \cdot \boldsymbol{Q}^T(t) = \underset{\substack{\boldsymbol{X}' \in \mathcal{B} \\ \tau^* \leq t^*}}{\mathcal{F}} [\boldsymbol{x}^*(\boldsymbol{X}', \tau^*), \boldsymbol{X}, t^*] \ . \tag{5.15}$$

for any arbitrary orthogonal tensor-valued functions of the time $\boldsymbol{Q}(t)$, any arbitrary vector-valued function $\boldsymbol{b}(t)$ and any arbitrary real number a.

Now, let us examine the restrictions imposed on the forms of the constitutive functional by considering separately three special rigid changes of frame, which taken successively in any order can represent the general rigid motion of the spatial frame of reference.

(a) *Shift of time* such that the present time t becomes the reference time:

$$\boldsymbol{Q}(\tau) = \boldsymbol{I} \ , \qquad \boldsymbol{b}(\tau) = 0 \ , \qquad \text{and} \qquad a = t \tag{5.16}$$

for all past times $\tau \leq t$, where t is the present time. Equation (5.9) gives $\boldsymbol{x}^*(\boldsymbol{X}, \tau^*) = \boldsymbol{x}(\boldsymbol{X}, \tau)$, and $t^* = 0$. Then

$$\mathcal{F} \ [\boldsymbol{x}(\boldsymbol{X}', \tau), \boldsymbol{X}, a] = \mathcal{F} \ [\boldsymbol{x}(\boldsymbol{X}', \tau), \boldsymbol{X}, 0] \ . \tag{5.17}$$

Thus $\mathcal{F}$ cannot depend *explicitly* on time.

(b) *Rigid translation of the spatial frame* such that the moving origin moves with the material point $\boldsymbol{X}$:

$$\boldsymbol{Q}(\tau) = \boldsymbol{I} \ , \qquad \boldsymbol{b}(\tau) = -\boldsymbol{x}(\boldsymbol{X}, \tau) \ , \qquad \text{and} \qquad a = 0 \ . \tag{5.18}$$

This means that the spatial frame of reference is translated so that the material point $\boldsymbol{X}$ at time τ remains at the origin. From (5.9) it follows that

$$\boldsymbol{x}^*(\boldsymbol{X}', \tau) = \boldsymbol{x}(\boldsymbol{X}', \tau) - \boldsymbol{x}(\boldsymbol{X}, \tau) \ , \qquad \tau^* = \tau \ , \tag{5.19}$$

Substituting this into (5.15) and using (5.7), we get

$$\boldsymbol{t}(\boldsymbol{X}, t) = \underset{\substack{\boldsymbol{X}' \in \mathcal{B} \\ \tau \leq t}}{\mathcal{F}} [\boldsymbol{x}(\boldsymbol{X}', \tau) - \boldsymbol{x}(\boldsymbol{X}, \tau), \boldsymbol{X}] \ . \tag{5.20}$$

Thus the stress at the material point $\boldsymbol{X}$ and time t depends only on the history for $\tau \leq t$ of the relative deformation (relative to $\boldsymbol{X}$) of the set of materials points $\boldsymbol{X}'$ in a neighborhood of $\boldsymbol{X}$.

According to the axiom of local action postulated at the beginning of this section, the neighborhood of material point $\boldsymbol{X}$, i.e., $|\boldsymbol{X}' - \boldsymbol{X}|$ is assumed to be arbitrarily small. Assuming differentiability, relative deformation may be permissible to approximate only by the first-order gradient

$$\boldsymbol{x}(\boldsymbol{X}', \tau) - \boldsymbol{x}(\boldsymbol{X}, \tau) \approx \boldsymbol{F}(\boldsymbol{X}, \tau) \cdot d\boldsymbol{X} \ , \tag{5.21}$$

where $\boldsymbol{F}(\boldsymbol{X}, \tau)$ is the deformation gradient tensor at $\boldsymbol{X}$ at time τ, and $d\boldsymbol{X} = \boldsymbol{X}' - \boldsymbol{X}$. This suggests that, since the relative motion history of an infinitesimal neighborhood of $\boldsymbol{X}$ is completely determined by the history of deformation gradient at $\boldsymbol{X}$, then the stress $\boldsymbol{t}(\boldsymbol{X}, t)$ must be determined by the history of $\boldsymbol{F}(\boldsymbol{X}, \tau)$ for $\tau \leq t$. Such materials are called *simple materials*. We also note that if we retain higher-order gradients in (5.21), then we obtain nonsimple materials of various classes. For example, by including the second-order gradients into argument of $\mathcal{F}$ we get the theory of *couple stress*. In other words, the behavior of the material point $\boldsymbol{X}$ is not affected by the histories of the distance points from $\boldsymbol{X}$. To any desired degree of accuracy, the whole configuration of a sufficiently small neighborhood of the material point $\boldsymbol{X}$ is determined by the history of the value of $\boldsymbol{F}(\boldsymbol{X}, \tau)$, and we may say that the stress $\boldsymbol{t}(\boldsymbol{X}, t)$, which was assumed to be determined by the local configuration, is completely determined by $\boldsymbol{F}(\boldsymbol{X}, \tau)$. That is, the the general constitutive equation (5.7) reduces to the form

$$\boldsymbol{t}(\boldsymbol{X}, t) = \underset{\tau \leq t}{\mathcal{F}} \left[\boldsymbol{F}(\boldsymbol{X}, \tau), \boldsymbol{X}\right] \ . \tag{5.22}$$

The materials described by this constitutive equation are memory-dependent, that is, the stress at $\boldsymbol{X}$ at time t depends on the history up to t of the two-point-tensor-valued function $\boldsymbol{F}$ of the simple argument τ (for fixed $\boldsymbol{X}$). For brevity of notation we omit writing $\boldsymbol{X}$ in the arguments of $\mathcal{F}$:

$$\boldsymbol{t}(t) = \underset{\tau \leq t}{\mathcal{F}} \left[\boldsymbol{F}(\tau)\right] \ . \tag{5.23}$$

(c) <u>*Time-dependent rigid rotations of the spatial frame of reference*</u>. Now, we consider the restrictions imposed on $\mathcal{F}$ by the principle of frame indifference under arbitrary time-dependent rotation such that $\boldsymbol{b}(\tau) = 0$, $a = 0$, and $\boldsymbol{Q}(\tau)$ is arbitrary. In this rotation, the stress tensor $\boldsymbol{t}$ transforms as

$$\boldsymbol{t}^*(t) = \boldsymbol{Q}(t) \cdot \boldsymbol{t}(t) \cdot \boldsymbol{Q}^T(t) \ , \tag{5.24}$$

where writing $\boldsymbol{X}$ in the argument of stress $\boldsymbol{t}$ is omitted. Using (5.23) and (5.12) we can conclude that the functional $\mathcal{F}$ must satisfy a restrictive condition of the form

$$\boldsymbol{Q}(t) \cdot \underset{\tau \leq t}{\mathcal{F}} \left[\boldsymbol{F}(\tau)\right] \cdot \boldsymbol{Q}^T(t) = \underset{\tau \leq t}{\mathcal{F}} \left[\boldsymbol{Q}(\tau) \cdot \boldsymbol{F}(\tau)\right] \tag{5.25}$$

for all orthogonal tensor-valued functions $\boldsymbol{Q}(\cdot)$ and all deformation processes $\boldsymbol{F}(\cdot)$. To solve this functional equation, we recall the polar decomposition (1.38) of the deformation gradient

$$\boldsymbol{F}(\tau) = \boldsymbol{R}(\tau) \cdot \boldsymbol{U}(\tau) \tag{5.26}$$

into a rotation tensor $\boldsymbol{R}$ and the right stretch tensor $\boldsymbol{U} = \sqrt{\boldsymbol{C}} = \sqrt{\boldsymbol{F}^T \cdot \boldsymbol{F}}$. Since the functional equation (5.25) is postulated to hold for all rotation histories $\boldsymbol{Q}(\cdot)$, we set $\boldsymbol{Q}(\tau) = \boldsymbol{R}^T(\tau)$ and arrive at

$$\boldsymbol{t}(t) = \boldsymbol{R}(t) \cdot \underset{\tau \le t}{\mathcal{F}} [\boldsymbol{U}(\tau)] \cdot \boldsymbol{R}^T(t) \ . \tag{5.27}$$

Thus the frame indifference requires according to eqn.(5.27) that the dependence of stress on $\boldsymbol{F}$ must take the form of an arbitrary function of $\boldsymbol{U}$ with the additional explicit dependence on $\boldsymbol{R}$ as shown.

Many other reduced forms are possible. For instance, expressing rotation $\boldsymbol{R}(t)$ through the deformation gradient, $\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$, leads to the following representation of the Cauchy stress tensor,

$$\boldsymbol{t}(t) = \boldsymbol{F}(t) \cdot \boldsymbol{U}^{-1}(t) \cdot \underset{\tau \le t}{\mathcal{F}} [\boldsymbol{U}(\tau)] \cdot \boldsymbol{U}^{-1}(t) \cdot \boldsymbol{F}^T(t) \ . \tag{5.28}$$

With

$$j = \det \boldsymbol{F} = \det (\boldsymbol{R} \cdot \boldsymbol{U}) = \det \boldsymbol{U} \ , \tag{5.29}$$

we have

$$\boldsymbol{t}(t) = \frac{1}{j} \boldsymbol{F}(t) \cdot \boldsymbol{U}^{-1}(t) \cdot (\det \boldsymbol{U}) \underset{\tau \le t}{\mathcal{F}} [\boldsymbol{U}(\tau)] \cdot \boldsymbol{U}^{-1}(t) \cdot \boldsymbol{F}^T(t) \ . \tag{5.30}$$

Now, if the stretch $\boldsymbol{U}$ is replaced by the Green deformation tensor,

$$\boldsymbol{U}(\tau) = \sqrt{\boldsymbol{C}(\tau)} \ , \qquad \tau \le t \ , \tag{5.31}$$

equation (5.30), after introducing a new functional $\tilde{\mathcal{F}}$ of the deformation history $\boldsymbol{C}$, can be written in the form

$$\boldsymbol{t}(t) = \frac{1}{j} \boldsymbol{F}(t) \cdot \underset{\tau \le t}{\tilde{\mathcal{F}}} [\boldsymbol{C}(\tau)] \cdot \boldsymbol{F}^T(t) \ . \tag{5.32}$$

Another useful reduced form may be obtained if the second Piola-Kirchhoff stress tensor $\boldsymbol{T}^{(2)}$ defined by eqn.(4.72),

$$\boldsymbol{T}^{(2)} = (\det \boldsymbol{F}) \, \boldsymbol{F}^{-1} \cdot \boldsymbol{t} \cdot (\boldsymbol{F}^T)^{-1} \ , \tag{5.33}$$

is used in the constitutive equation instead of the Cauchy stress tensor $\boldsymbol{t}$. With (5.28), we obtain

$$\boldsymbol{T}^{(2)}(t) = (\det \boldsymbol{U}(t))\, \boldsymbol{U}^{-1}(t) \cdot \underset{\tau \leq t}{\mathcal{F}} \left[\boldsymbol{U}(\tau)\right] \cdot \boldsymbol{U}^{-1}(t) \ , \tag{5.34}$$

and we can see that the right-hand side is a functional of the stretch history $\boldsymbol{U}(\cdot)$ only. Now, we replace the stretch $\boldsymbol{U}$ by the Green deformation tensor according to (5.31), substitute this into eqn.(5.34), and define a new functional $\mathcal{G}$ of the deformation history $\boldsymbol{C}$. The general mechanical constitutive equation can be written in the reduced form as

$$\boldsymbol{T}^{(2)}(t) = \underset{\tau \leq t}{\mathcal{G}} \left[\boldsymbol{C}(\tau)\right] \ . \tag{5.35}$$

According to this result, the second Piola-Kirchhoff stress tensor is a function of the past history of the Green deformation tensor. In view of (1.31) and (5.12), we have

$$\boldsymbol{C}^* = (\boldsymbol{F}^*)^T \cdot \boldsymbol{F}^* = \boldsymbol{F}^T \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q} \cdot \boldsymbol{F} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{C} \ . \tag{5.36}$$

It is then easy to verify the identity

$$\left(\boldsymbol{T}^{(2)}\right)^* = \mathcal{G}\left(\boldsymbol{C}^*\right) = \mathcal{G}\left(\boldsymbol{C}\right) = \boldsymbol{T}^{(2)} \ . \tag{5.37}$$

Therefore the reduced form (5.35) is necessary and sufficient to satisfy the axiom of frame indifference. In other words, this axiom has led to the conclusion that in terms of the second Piola-Kirchhoff stress tensor the response functional $\mathcal{G}$ depends only on the scalars $C_{KL} = \boldsymbol{x}_{,K} \cdot \boldsymbol{x}_{,L}$ and not on the rotation. We often say that the response functional $\mathcal{G}$ is an isotropic function of three vectors $\boldsymbol{x}_{,K}$.

Another useful reduced form, equivalent to (5.35), arises if we express the past strain history $\boldsymbol{C}(\tau) = \boldsymbol{C}(t-s)$, $(\tau \leq t,\, s \geq 0)$ in terms of the relative difference history $\boldsymbol{C}_d^t(s)$,

$$\boldsymbol{C}_d^t(s) := \boldsymbol{C}(t-s) - \boldsymbol{C}(t) \ . \tag{5.38}$$

We insert $\boldsymbol{C}(\tau) = \boldsymbol{C}(t) + \boldsymbol{C}_d^t(s)$ in (5.35) and obtain

$$\boldsymbol{T}^{(2)}(t) = \underset{\tau \leq t}{\mathcal{G}} \left[\boldsymbol{C}(\tau)\right] = \underset{s \geq 0}{\mathcal{H}} \left[\boldsymbol{C}_d^t(s); \boldsymbol{C}(t)\right] \ , \tag{5.39}$$

where $\mathcal{H}$ is a new functional of the difference history $\boldsymbol{C}_d^t(\cdot)$ depending on the present strain $\boldsymbol{C}(t)$ as a parameter. If we introduce the *static* part

$$\boldsymbol{f}(\boldsymbol{C}) := \underset{s \geq 0}{\mathcal{H}} \left[\boldsymbol{0}(s); \boldsymbol{C}(t)\right] \tag{5.40}$$

and the *memory* part

$$\underset{s \geq 0}{\mathcal{F}} \left[\boldsymbol{C}_d^t(s); \boldsymbol{C}(t)\right] := \underset{s \geq 0}{\mathcal{H}} \left[\boldsymbol{C}_d^t(s); \boldsymbol{C}(t)\right] - \boldsymbol{f}(\boldsymbol{C}) \; , \tag{5.41}$$

we may deduce a second version of the reduced form (5.39),

$$\boldsymbol{T}^{(2)}(t) = \boldsymbol{f}(\boldsymbol{C}(t)) + \underset{s \geq 0}{\mathcal{F}} \left[\boldsymbol{C}_d^t(s); \boldsymbol{C}(t)\right] \; . \tag{5.42}$$

According to this equation the second Piola-Kirchhoff stress tensor is decomposed into a static part and a memory part. The static part is a function of the present static strain $\boldsymbol{C}(t)$, whereas the memory part is a function of the past history of the difference strain $\boldsymbol{C}_d^t(\cdot)$ depending on the present strain $\boldsymbol{C}(t)$ as a parameter. In particular, the memory part vanishes for strain history which is identically constant ('static history'), i.e.,

$$\underset{s \geq 0}{\mathcal{F}} \left[\mathbf{0}(s); \boldsymbol{C}(t)\right] = 0 \; . \tag{5.43}$$

We close this section with a brief interpretation of the underlying ideas. The functional formulations of the most general mechanical constitutive equation (5.27) or (5.32) or (5.35) or (5.39) corresponds to the assumption that a material body is able to memorize past events of its deformation. If there is no memory of the material, the memory functional $\mathcal{F}$ in the reduced form (5.42) degenerates to zero and

$$\boldsymbol{T}^{(2)}(t) = \boldsymbol{f}(\boldsymbol{C}(t)) \; . \tag{5.44}$$

In this case, a material body is called *elastic*.

Conversely, any kind of *inelastic* behavior corresponds to a memory property of the material. The specific memory behavior of a particular inelastic material can be very different in its intensity and character. It may be a long-range memory, as it is usually the case in *plasticity*. The material memory can fade away for those events which are long ago; this character is associated to the term *viscoelasticity*. In some cases it may happen that only an infinitesimal part of the strain history influences the present stress. Then the stress tensor depends on strain rate and the term *viscosity* is the appropriate characterization.

The general theory of material behavior provides a large variety of possibilities to obtain special representation of the functional $\mathcal{F}$ or $\tilde{\mathcal{F}}$ or $\mathcal{G}$ or $\mathcal{H}$ in the reduced forms (5.27) or (5.32) or (5.35) or (5.39). Functional relations can be defined *explicitly* by integrals or be tensor-valued functions of strain and strain-rate tensors. They can also be defined *implicitly* by means of ordinary differential equations.

5.3 Elastic materials

Definition. *A simple material is called elastic if the stress t_{kl}, heat vector q_k, internal energy density ε, and entropy density η at $(\boldsymbol{X}, t)$ depend only on the deformation gradient $\boldsymbol{x}_{,K}$ and temperature θ, not on the entire past thermomechanical history.*

This definition of the elastic solid can be expressed mathematically by posing a set of constitutive equations of the form

$$
\begin{aligned}
t_{kl} &= f_{kl}(\boldsymbol{x}_{,K}, \theta, \boldsymbol{X}, t) \ , \\
q_k &= g_k(\boldsymbol{x}_{,K}, \theta, \boldsymbol{X}, t) \ , \\
\varepsilon &= e(\boldsymbol{x}_{,K}, \theta, \boldsymbol{X}, t) \ , \\
\eta &= n(\boldsymbol{x}_{,K}, \theta, \boldsymbol{X}, t) \ .
\end{aligned}
\tag{5.45}
$$

The axiom of the objectivity applied to the first constitutive equation implies that the response function f_{kl} cannot explicitly depend on time t and the dependence of f_{kl} on the deformation gradient must take a special explicit form. Here, we start with the reduced form (5.32); its componental form reads

$$
t_{kl} = \frac{\varrho}{\varrho_0} F_{KL}(\boldsymbol{C}, \theta, \boldsymbol{X})\, x_{k,K} x_{l,L} \ , \tag{5.46}
$$

where

$$
F_{KL}(\boldsymbol{C}, \theta, \boldsymbol{X}) := \boldsymbol{I}_K \cdot \tilde{\mathcal{F}}(\boldsymbol{C}, \theta, \boldsymbol{X}) \cdot \boldsymbol{I}_L \ . \tag{5.47}
$$

The axiom of objectivity can be applied analogously to vector and scalar-valued functions (such as g_k, e and n) resulting in

$$
\begin{aligned}
q_k &= \frac{\varrho}{\varrho_0} G_K(\boldsymbol{C}, \theta, \boldsymbol{X})\, x_{k,K} \ , \\
\varepsilon &= E(\boldsymbol{C}, \theta, \boldsymbol{X}) \ , \\
\eta &= N(\boldsymbol{C}, \theta, \boldsymbol{X}) \ .
\end{aligned}
\tag{5.48}
$$

Next, we employ the Clausius-Duhem inequality (5.5). Differentiating $(5.48)_{2,3}$ with respect to time,

$$
\begin{aligned}
\dot{\varepsilon} &= \frac{\partial E}{\partial C_{KL}} \dot{C}_{KL} + \frac{\partial E}{\partial \theta} \dot{\theta} \ , \\
\dot{\eta} &= \frac{\partial N}{\partial C_{KL}} \dot{C}_{KL} + \frac{\partial N}{\partial \theta} \dot{\theta} \ ,
\end{aligned}
$$

and substituting these into (5.5), we get

$$
\varrho \left(\frac{\partial N}{\partial C_{KL}} - \frac{1}{\theta} \frac{\partial E}{\partial C_{KL}} \right) \dot{C}_{KL} + \varrho \left(\frac{\partial N}{\partial \theta} - \frac{1}{\theta} \frac{\partial E}{\partial \theta} \right) \dot{\theta} + \frac{1}{\theta} \frac{\varrho}{\varrho_0} F_{KL} x_{k,K} x_{l,L} d_{lk} + \frac{1}{\theta^2} \operatorname{grad} \theta \cdot \boldsymbol{q} \geq 0 \ . \tag{5.49}
$$

From (2.12), we have

$$
\dot{C}_{KL} = 2 d_{kl} x_{k,K} x_{l,L} \ ,
$$

so that,

$$
\varrho \left(\frac{\partial N}{\partial C_{KL}} - \frac{1}{\theta} \frac{\partial E}{\partial C_{KL}} + \frac{1}{2\varrho_0 \theta} F_{KL} \right) \dot{C}_{KL} + \varrho \left(\frac{\partial N}{\partial \theta} - \frac{1}{\theta} \frac{\partial E}{\partial \theta} \right) \dot{\theta} + \frac{1}{\theta^2} \operatorname{grad} \theta \cdot \boldsymbol{q} \geq 0 \ . \tag{5.50}
$$

This inequality must be satisfied for all independent thermomechanical processes according to the so-called **axiom of thermomechanical admissibility**. Since $\dot{C}_{KL}$, $\dot{\theta}$, and $\operatorname{grad}\theta$ occur only linearly with coefficients which are not functions of these quantities, (5.50) cannot be maintained for all $\dot{C}_{KL}$, $\dot{\theta}$, and $\operatorname{grad}\theta$ unless the coefficients of these terms vanish separately. Thus

$$F_{KL} = 2\varrho_0 \left(\frac{\partial E}{\partial C_{KL}} - \theta \frac{\partial N}{\partial C_{KL}} \right) , \tag{5.51}$$

$$0 = \frac{\partial N}{\partial \theta} - \frac{1}{\theta}\frac{\partial E}{\partial \theta} , \tag{5.52}$$

and

$$\operatorname{grad}\theta \neq 0 \quad \text{and} \quad \boldsymbol{q} = 0 \qquad \text{for } \textit{adiabatic} \text{ changes} , \tag{5.53}$$

or

$$\operatorname{grad}\theta = 0 \quad \text{and} \quad \boldsymbol{q} \neq 0 \qquad \text{for } \textit{isothermal} \text{ changes} . \tag{5.54}$$

In place of the internal energy density ε, it is convenient to introduce the *Helmholtz free energy* ψ by

$$\psi := \varepsilon - \theta\eta = E - \theta N = \psi(\boldsymbol{C}, \theta, \boldsymbol{X}) . \tag{5.55}$$

Using this in the above equations, we get

$$F_{KL} = 2\varrho_0 \frac{\partial \psi}{\partial C_{KL}} , \tag{5.56}$$

$$N = -\frac{\partial \psi}{\partial \theta} . \tag{5.57}$$

Substituting (5.56) and (5.57) into (5.46) and $(5.48)_3$, we get

$$t_{kl} = 2\varrho \frac{\partial \psi}{\partial C_{KL}} x_{k,K} x_{l,L} , \tag{5.58}$$

$$\eta = -\frac{\partial \psi}{\partial \theta} . \tag{5.59}$$

To ensure the symmetry of the tensor t_{kl}, we must take the symmetric part of tensor $x_{k,K}x_{l,L}$ on the right-hand side of the constitutive equation (5.58), so that,

$$t_{kl} = \varrho \frac{\partial \psi}{\partial C_{KL}} \left(x_{k,K} x_{l,L} + x_{l,K} x_{k,L} \right) . \tag{5.60}$$

We have therefore proved

THEOREM. *An elastic solid is thermodynamically admissible if and only if the stress, internal energy, and entropy are derivable from a potential W and the heat flux is zero (the solid must undergo locally adiabatic changes) or the temperature is constant (the solid must undergo locally isothermal changes)* so that

$$t_{kl} = \frac{\varrho}{\varrho_0} \frac{\partial W}{\partial C_{KL}} \left(x_{k,K} x_{l,L} + x_{l,K} x_{k,L} \right) . \tag{5.61}$$

$$\varepsilon = \frac{1}{\varrho_0}\left(W - \theta\,\frac{\partial W}{\partial \theta}\right) \ , \tag{5.62}$$

$$\eta = -\frac{1}{\varrho_0}\frac{\partial W}{\partial \theta} \ , \tag{5.63}$$

where

$$W(\boldsymbol{C}, \theta, \boldsymbol{X}) := \varrho_0 \psi \ . \tag{5.64}$$

Finally, we write the expressions for the Piola-Kirchhoff tensors T_{Kl} and T_{KL}. Substituting (5.61) into (4.51) and using the symmetry $C_{KL} = C_{LK}$, we get

$$T_{KL} = 2\,\frac{\partial W}{\partial C_{KL}} \ , \tag{5.65}$$

$$T_{Kl} = 2\,\frac{\partial W}{\partial C_{KM}}\,x_{l,M} \ . \tag{5.66}$$

5.3.1 Incompressible elastic solids

For the incompressible elastic solids, the stress constitutive equation (5.60) must be further restricted, since in this case the volume is preserved during the motion. It also means that the mass density remains unchanged, that is,

$$j = \sqrt{\det \boldsymbol{C}} = \frac{\varrho_0}{\varrho} = 1 \ , \tag{5.67}$$

where the first equality follows from $(1.31)_2$. This condition places a restriction on $\boldsymbol{C}$, namely, all components C_{KL} of $\boldsymbol{C}$ are not independent. Hence we must take proper caution in evaluating the partial derivatives $\partial\psi/C_{KL}$ occurring in (5.60). To take care of this situation, we employ the method of Lagrange's multipliers. Thus we replace the free energy $\psi = \psi(\boldsymbol{C}, \theta, \boldsymbol{X})$ by function

$$\tilde{\psi}(\boldsymbol{C}, \theta, \boldsymbol{X}, p) = \psi(\boldsymbol{C}, \theta, \boldsymbol{X}) - \frac{p}{2\varrho_0}\left(\det \boldsymbol{C} - 1\right) \ , \tag{5.68}$$

where p is the unknown Lagrange multiplier. Taking the partial derivative of $\tilde{\psi}$ with respect to C_{KL}, we obtain

$$\frac{\partial \tilde{\psi}}{\partial C_{KL}} = \frac{\partial \psi}{\partial C_{KL}} - \frac{p}{2\varrho_0}\frac{\partial \det \boldsymbol{C}}{\partial C_{KL}} \ ,$$

where $\partial\psi/\partial C_{KL}$ is to be calculated without any regard to the constraint (5.67). Combining (1.25) and $(1.33)_2$, we get

$$\frac{\partial \det \boldsymbol{C}}{\partial C_{KL}} = (\det \boldsymbol{C})\,B_{LK} = (\det \boldsymbol{C})\,X_{K,m}X_{L,m} \tag{5.69}$$

where we have substituted for the Piola deformation tensor $\boldsymbol{B}$ from $(1.32)_2$. Substituting the last relations and using the constraint (5.69), (5.60) takes the form

$$t_{kl} = -p\delta_{kl} + \varrho \frac{\partial \psi}{\partial C_{KL}} \left(x_{k,K} x_{l,L} + x_{l,K} x_{k,L} \right) \ . \tag{5.70}$$

The function $p(\boldsymbol{x}, t)$ introduced here is called *pressure*. Equation (5.70) shows that in the case of incompressibility the constitutive equation determines the stress tensor up to a pressure. This unknown is to be determined upon integration of differential equations and using the boundary conditions. We caution the reader of the difference between this pressure and the thermodynamic pressure.

5.3.2 Linear elastic materials

In the preceding sections we found that the most general constitutive equations of an elastic solid have the forms expressed by equations (5.61) to (5.64), where (5.61) can be expressed alternatively by (5.65) or (5.66) when the Lagrangian representation is considered. For our purpose, we write these equations in terms of the Lagrangian strain measure $\boldsymbol{E}$ related to $\boldsymbol{C}$ by

$$E_{KL} := \frac{1}{2} \left(C_{KL} - \delta_{KL} \right) \ . \tag{5.71}$$

Equation (5.65) in terms of $\boldsymbol{E}$ reads

$$T_{KL} = \frac{\partial W}{\partial E_{KL}} \ , \tag{5.72}$$

where the potential W is now to be considered a function of $\boldsymbol{E}$, θ, and $\boldsymbol{X}$, that is,

$$W = W(\boldsymbol{E}, \theta, \boldsymbol{X}) = \varrho_0 \psi \ , \tag{5.73}$$

where ψ is the free energy.

By using the equation (5.72) of nonlinear elastic solids, we can derive various approximate theories. A polynomial approximation in the strain components E_{KL} is based upon an expansion of the form

$$W = W_0 + W_{KL} E_{KL} + \frac{1}{2} W_{KLMN} E_{KL} E_{MN} + W_{KLMNPQ} E_{KL} E_{MN} E_{PQ} \cdots \ , \tag{5.74}$$

where W_0, W_{KL}, W_{KLMN}, W_{KLMNPQ} are, in general, functions of $\boldsymbol{X}$, for inhomogeneous solids, and θ. For the homogeneous solids they are functions of θ only.

We are interested only in the linear theory which assumes that *either* $E_{KL} \ll 1$ and the strain measure $\boldsymbol{E}$ can be replaced by the infinitesimal strain tensor $\tilde{\boldsymbol{E}}$ *or* that $|W_{KLMNPQ}| \ll |W_{KLMN}|$ and $|W_{KLMNPQ}| \ll |W_{KL}|$. Thus, we need not retain terms beyond the quadratic terms in $\boldsymbol{E}$ in (5.74):

$$W = W_0 + W_{KL} E_{KL} + \frac{1}{2} W_{KLMN} E_{KL} E_{MN} \ . \tag{5.75}$$

Substituting (5.75) into (5.72), we get

$$T_{KL} = W_{KL} + \frac{1}{2}\left(W_{KLMN} + W_{MNKL}\right) E_{MN} \ . \tag{5.76}$$

If the stress vanishes in the natural state, then $W_{KL} = 0$. Introducing

$$C_{KLMN} := \frac{1}{2}\left(W_{KLMN} + W_{MNKL}\right) \ , \tag{5.77}$$

the constitutive equation (5.76) reduces to the form

$$T_{KL} = C_{KLMN} E_{MN} \ . \tag{5.78}$$

where the *elastic coefficients* C_{KLMN} may, in general, depend upon the position $\boldsymbol{X}$ and temperature θ. If the elastic coefficients are constant, the material is said to be *homogeneous.* The constitutive law given by (5.78) is known as the *generalized Hooke's law.*

The definition (5.77) shows that the elastic coefficients are symmetric with respect to KL and MN:

$$C_{KLMN} = C_{MNKL} \ . \tag{5.79}$$

Furthermore, the symmetry of both the stress and strain tensors implies that

$$C_{KLMN} = C_{LKMN} = C_{KLNM} \ . \tag{5.80}$$

Thus, the symmetry properties (5.79) and (5.80) reduce $3^4 = 81$ independent elastic coefficients to 21 distinct coefficients at most. Finally, the stress potential can be expressed as

$$W = \frac{1}{2} T_{KL} E_{KL} \ . \tag{5.81}$$

In the *linearized theory* we assume that displacement gradients are everywhere much smaller than unity, i.e., $\partial u_k / \partial x_l \ll 1$. Under such assumption, the strain tensor E_{KL} can be replaced by the infinitesimal strain tensor $\tilde{E}_{KL}$, $E_{KL} \cong \tilde{E}_{KL}$. Moreover, the linearization also means that no distinction need be made between the reference coordinates X_K and the current spatial position x_k of the same material point; the shifter symbol δ_{Kk} is reduced to the Kronecker delta. If we substitute

$$\frac{1}{j} = 1 - u_{k,k} \ , \qquad x_{k,K} = \delta_{Kk} + u_{k,K} \ , \tag{5.82}$$

and (5.78) into $(4.50)_2$, we get

$$t_{kl} = \left(1 - u_{k,k}\right)\left(\delta_{Kk} + u_{k,K}\right)\left(\delta_{Ll} + u_{l,L}\right) C_{KLMN} \tilde{E}_{MN} \ . \tag{5.83}$$

Remembering that for small strains

$$\tilde{E}_{MN} \cong \tilde{e}_{mn} \delta_{Mm} \delta_{Nn} \ , \tag{5.84}$$

we may linearize (5.83) as

$$t_{kl} = c_{klmn}\tilde{e}_{mn} \ , \qquad \text{or} \qquad \boldsymbol{t} = \boldsymbol{c} : \tilde{\boldsymbol{e}} \ , \tag{5.85}$$

where

$$c_{klmn} := C_{KLMN}\delta_{Kk}\delta_{Ll}\delta_{Mm}\delta_{Nn} \tag{5.86}$$

are the spatial elastic moduli that are subject to the symmetry conditions

$$c_{klmn} = c_{mnkl} = c_{lkmn} = c_{klnm} \ . \tag{5.87}$$

The stress potential can now be expressed as

$$W = \frac{1}{2}c_{klmn}\tilde{e}_{kl}\tilde{e}_{mn} \ . \tag{5.88}$$

5.3.3 Hooke's law for isotropic media

The highest symmetry of a material is reached if a solid possesses no preferred direction with respect to its elastic property. This also means that the elastic tensor c_{klmn} is invariant under any orthogonal transformation of the coordinate system. In such a case, the material is said to be *isotropic*. Otherwise, the material is *anisotropic*.

Note that any orthogonal transformation of the coordinate system may be expressed by transformation equations

$$x'_k = Q_{kl}x_l \tag{5.89}$$

subject to

$$Q_{kl}Q_{ml} = Q_{lk}Q_{lm} = \delta_{km} \ , \qquad \det Q_{kl} = \pm 1 \ . \tag{5.90}$$

The components of a second-order Cartesian tensor $\boldsymbol{t}$ transform under the orthogonal transformation of the coordinate system as

$$t'_{kl} = Q_{km}Q_{ln}t_{mn} \ , \tag{5.91}$$

which may be readily inverted with the help of the orthogonality conditions to yield

$$t_{kl} = Q_{mk}Q_{nl}t'_{mn} \ . \tag{5.92}$$

Note carefully the location of the summed indices m and n in (5.91) and (5.92).

The generalized Hooke's law for isotropic material is reduced to the form

$$t_{kl} = \lambda\vartheta\delta_{kl} + 2\mu\tilde{e}_{kl} \ , \qquad \text{or} \qquad \boldsymbol{t} = \lambda\vartheta\boldsymbol{I} + 2\mu\tilde{\boldsymbol{e}} \ , \tag{5.93}$$

where

$$\vartheta := \tilde{e}_{kk} = \operatorname{div}\boldsymbol{u} \ . \tag{5.94}$$

We see that for isotropic elastic behavior the 21 constants of the generalized Hooke's law are reduced to two, λ and μ, known as the *Lamé coefficients*.

Proof of (5.93) will be accomplished in three steps. First, we show that *for an isotropic linear elastic solid the principal axes of the stress and (small) strain tensors coincide.* To proof it, we take, without loss of generality, the coordinate axes x_k in the principal directions of strain tensor $\boldsymbol{e}$. Then $\tilde{e}_{12} = \tilde{e}_{13} = \tilde{e}_{23} = 0$. We shall now show that $t_{23} = 0$. We first have

$$t_{23} = A\tilde{e}_{11} + B\tilde{e}_{22} + C\tilde{e}_{33}$$

with $A := c_{2311}$, $B := c_{2322}$, and $C := c_{2333}$. We now rotate the coordinate system through an angle of 180^o about the x_3-axis. Then $x'_1 = -x_1$, $x'_2 = -x_2$, and $x'_3 = x_3$, and the matrix of the transformation

$$\boldsymbol{x}' = \boldsymbol{Q}\boldsymbol{x}$$

is

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

In view of (5.91), we therefore have

$$t'_{23} = Q_{2m}Q_{3n}t_{mn} = -t_{23} ,$$

$$\tilde{e}'_{11} = Q_{1m}Q_{1n}\tilde{e}_{mn} = \tilde{e}_{11} ,$$

$$\tilde{e}'_{22} = Q_{2m}Q_{2n}\tilde{e}_{mn} = \tilde{e}_{22} ,$$

$$\tilde{e}'_{33} = \tilde{e}_{33} .$$

The relation

$$t'_{23} = A\tilde{e}'_{11} + B\tilde{e}'_{22} + C\tilde{e}'_{33} = A\tilde{e}_{11} + B\tilde{e}_{22} + C\tilde{e}_{33} = t_{23}$$

is now the consequence of isotropy since the constants A, B, and C do not depend on the coordinate system. Thus

$$-t_{23} = t'_{23} = t_{23} ,$$

which implies that $t_{23} = 0$. Similarly it can be shown that $t_{12} = t_{13} = 0$.

Second, consider the component t_{11}. Taking the coordinate axes in principal directions of strain, we obtain

$$t_{11} = a\tilde{e}_{11} + b\tilde{e}_{22} + c\tilde{e}_{33} ,$$

with $a := c_{1111}$, $b := c_{1122}$, and $c := c_{1133}$. We now rotate the coordinate system through an angle 90^o about the x_1-axis in such a way that

$$x'_1 = x_1 , \quad x'_2 = x_3 , \quad x'_3 = -x_2 .$$

The matrix of this transformation is

$$\boldsymbol{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} .$$

We have

$$t'_{11} = Q_{1m}Q_{1n}t_{mn} = t_{11} \ ,$$

$$\tilde{e}'_{11} = \tilde{e}_{11} \ , \quad \tilde{e}'_{22} = Q_{2m}Q_{2n}\tilde{e}_{mn} = \tilde{e}_{33} \ , \quad \tilde{e}'_{33} = \tilde{e}_{22} \ .$$

In view of isotropy, the constants a, b, and c do not depend on the coordinate system, so that

$$t'_{11} = a\tilde{e}'_{11} + b\tilde{e}'_{22} + c\tilde{e}'_{33} \ ,$$

and substituting for $\tilde{e}'_{kk}$, we get

$$t'_{11} = a\tilde{e}_{11} + b\tilde{e}_{33} + c\tilde{e}_{22} \ .$$

This implies that $b = c$ since $t_{11} = a\tilde{e}_{11} + b\tilde{e}_{22} + c\tilde{e}_{33}$. We can thus write t_{11} as

$$t_{11} = a\tilde{e}_{11} + b\left(\tilde{e}_{22} + \tilde{e}_{33}\right) = \lambda\vartheta + 2\mu\tilde{e}_{11} \ ,$$

where

$$\lambda = b = c_{1122} \ , \quad 2\mu = a - b = c_{1111} - c_{1122} \ .$$

The same relations can be obtained for subscripts 2 and 3. In summary, we have

$$t_{11} = \lambda\vartheta + 2\mu\tilde{e}_{11} \ ,$$

$$t_{22} = \lambda\vartheta + 2\mu\tilde{e}_{22} \ ,$$

$$t_{33} = \lambda\vartheta + 2\mu\tilde{e}_{33} \ ,$$

$$t_{kl} = 0 \quad \text{for} \quad k \neq j \ ,$$

which is the generalized Hooke's law for isotropic body in principal directions. Shortly written,

$$t_{kl} = \lambda\vartheta\delta_{kl} + 2\mu\tilde{e}_{kl} \ .$$

Third, we now rotate the coordinate system with the axes x_k coinciding with principal directions of strain to arbitrary coordinate system with the axes x'_k and show that Hooke's law for isotropic material also holds in a rotated coordinate system x'_k. Denoting by Q_{kl} the transformation matrix of this rotation, the stress and strain tensor transform according to the transformation law (5.91) for second-order tensors. Multiplying the above equation by $Q_{mk}Q_{nl}$, using the transformation law (5.91) and the orthogonality property (5.90) of Q_{kl}, we have

$$t'_{mn} = \lambda\vartheta\delta_{mn} + 2\mu\tilde{e}'_{mn} \ .$$

Moreover,

$$\vartheta' = \tilde{e}'_{mm} = Q_{mk}Q_{ml}\tilde{e}_{kl} = \delta_{kl}\tilde{e}_{kl} = \tilde{e}_{kk} = \vartheta \ .$$

We finally have

$$t'_{mn} = \lambda\vartheta'\delta_{mn} + 2\mu\tilde{e}'_{mn} \ ,$$

which proves (5.93).

We have seen that the Lamé coefficients are expressed in terms of the elastic coefficients as

$$\lambda = c_{1122} \; , \qquad \mu = c_{1212} \; . \tag{5.95}$$

We note without proof that the most general fourth-order isotropic tensor $\boldsymbol{s}$ is of the form

$$s_{klmn} = \lambda \delta_{kl} \delta_{mn} + \mu \left(\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm} \right) + \nu \left(\delta_{km} \delta_{ln} - \delta_{kn} \delta_{lm} \right) \; , \tag{5.96}$$

where λ, μ, and ν are scalars. If $\boldsymbol{s}$ is replaced by the elastic tensor $\boldsymbol{c}$, the symmetry relations $c_{klmn} = c_{lkmn} = c_{klnm}$ imply that ν must be zero since by interchanging k and l in the expression

$$\nu \left(\delta_{km} \delta_{ln} - \delta_{kn} \delta_{lm} \right) = \nu \left(\delta_{lm} \delta_{kn} - \delta_{ln} \delta_{km} \right)$$

we see that $\nu = -\nu$ and, consequently, $\nu = 0$. Thus, the elastic tensor for a linear, isotropic solid reads

$$c_{klmn} = \lambda \delta_{kl} \delta_{mn} + \mu \left(\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm} \right) \; . \tag{5.97}$$

5.3.4 Restrictions on elastic coefficients

In this section, we analyze several hypothetical experiments and consequent restrictions that must be placed upon elastic moduli in order that they may represent a real material adequately.

We will assume that Hooke's law (5.93) for linear isotropic solid is invertible for $\tilde{e}_{kl}$. In fact, setting $k = l$, this equation gives

$$t_{kk} = (3\lambda + 2\mu) \, \tilde{e}_{kk} \; . \tag{5.98}$$

Now, by solving (5.93) for $\tilde{e}_{kl}$ and substituting from (5.98), we obtain the inverse form of Hooke's law for the isotropic material,

$$\tilde{e}_{kl} = \frac{1}{2\mu} t_{kl} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \, t_{mm} \delta_{kl} \; . \tag{5.99}$$

We observe that for $\tilde{e}_{kl}$ *to be uniquely determined by* t_{kl} *we must have*

$$\mu \neq 0 \; , \qquad 3\lambda + 2\mu \neq 0 \tag{5.100}$$

and *in order not to have zero strain for a finite stress we must have*

$$|\mu| < \infty \; , \qquad |3\lambda + 2\mu| < \infty \; . \tag{5.101}$$

Let us decompose the infinitesimal strain tensor $\tilde{\boldsymbol{e}}$ into two component tensors:

$$\tilde{e}_{kl} = \frac{1}{3} \vartheta \delta_{kl} + \tilde{\eta}_{kl} \; , \tag{5.102}$$

where the so-called *spherical strain tensor* $\vartheta\delta_{kl}/3$ is proportional to the unit second-order tensor and factor ϑ, which is defined by (5.94) and known as the *mean normal strain*, and the so-called *deviator strain tensor* $\tilde{\eta}_{kl}$ is a trace-free, symmetric, second-order tensor,

$$\tilde{\eta}_{kl} := \tilde{e}_{kl} - \frac{1}{3}\vartheta\delta_{kl} \ . \tag{5.103}$$

The same decomposition of the stress tensor $\boldsymbol{t}$ reads

$$t_{kl} = -p\delta_{kl} + \xi_{kl} \ , \tag{5.104}$$

where the scalar p is the negative of the mean normal stress and called *mechanical pressure*,

$$p := -\frac{1}{3}\,t_{mm} \ , \tag{5.105}$$

and the *deviator stress tensor* ξ_{kl} is a trace-free, symmetric, second-order tensor, defined by

$$\xi_{kl} := t_{kl} + p\delta_{kl} \ . \tag{5.106}$$

The isotropic Hooke's law takes a particularly simple form in spherical and deviator parts of $\tilde{\boldsymbol{e}}$ and $\boldsymbol{t}$. Substituting (5.94) and (5.105) into (5.98) gives,

$$-\,p = k\vartheta \ , \tag{5.107}$$

with

$$k := \lambda + \frac{2}{3}\mu \tag{5.108}$$

referred to as the *elastic bulk modulus.* To find the equation connecting the deviatoric parts, we substitute (5.102) and (5.104) into (5.93) and use (5.107):

$$\xi_{kl} = 2\mu\tilde{\eta}_{kl} \ , \tag{5.109}$$

where μ is also called the *elastic shear modulus.*

Finally, we derive the strain-energy function for an isotropic elastic solid. For this purpose, we successively substitute (5.97) into (5.88), giving

$$W(\tilde{e}_{kl}) = \frac{1}{2}\lambda\vartheta^2 + \mu\tilde{e}_{kl}\tilde{e}_{kl} \ , \tag{5.110}$$

which, using (5.103) and (5.108), can be rewritten as

$$W(\tilde{e}_{kl}) = \frac{1}{2}k\vartheta^2 + \mu\tilde{\eta}_{kl}\tilde{\eta}_{kl} \ . \tag{5.111}$$

(i) Hydrostatic pressure. Experimental observations indicate that under hydrostatic pressure the volume of an elastic solid diminishes. The state of stress at a point of the body is said to be hydrostatic if the stress tensor has the form

$$t_{kl} = -p\delta_{kl} \ , \qquad p > 0 \ . \tag{5.112}$$

From (5.107) we see that

$$p = -k\vartheta \ , \tag{5.113}$$

where $\vartheta := \tilde{e}_{kk} = \operatorname{div} \boldsymbol{u} = j - 1 = (dv - dV)/dV$ is the cubical dilatation. In the case of hydrostatic pressure, clearly $dv < dV$ and hence $\vartheta < 0$. We must thus have $k \geq 0$ which implies that

$$3\lambda + 2\mu \geq 0 \ . \tag{5.114}$$

(ii) Simple shear. Consider a simple constant shear in which

$$t_{12} \neq 0 \ , \qquad t_{kl} = 0 \quad \text{otherwise} \ . \tag{5.115}$$

In this case (5.99) gives

$$t_{12} = 2\mu\tilde{e}_{12} \ . \tag{5.116}$$

Experimental observation of small deformations of elastic solids subjected to simple shear indicates that t_{12} and $\tilde{e}_{12}$ have the same direction. Consequently,

$$\mu > 0 \ . \tag{5.117}$$

(iii) Uniaxial tension. Let a circular cylinder be subjected to a uniform axial tension t_{11} and all other $t_{kl} = 0$. Through (5.99) for this case we find that

$$\tilde{e}_{11} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} t_{11} \ , \qquad \tilde{e}_{22} = \tilde{e}_{33} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} t_{11} \ , \tag{5.118}$$

$$\tilde{e}_{23} = \tilde{e}_{31} = \tilde{e}_{12} = 0 \ .$$

Thus for $t_{11} > 0$ ($t_{11} < 0$), cylinder will elongate, $\tilde{e}_{11} > 0$ (shorten, $\tilde{e}_{11} < 0$) and its diameter will contract, $\tilde{e}_{22} < 0$, $\tilde{e}_{33} < 0$ (expand, $\tilde{e}_{22} > 0$, $\tilde{e}_{33} > 0$). Employing (5.114) and (5.117), this results in

$$\lambda > 0 \ . \tag{5.119}$$

Collecting the results of (i)–(iii) we have

$$0 < \lambda < \infty \ , \qquad 0 < \mu < \infty \ . \tag{5.120}$$

These restrictions on the moduli of isotropic materials are universally agreed.

Literature

Biot, M.A. (1965). *Mechanics of Incremental Deformations*, J. Wiley, New York.

Brdička, M. (1959). *Continuum Mechanics*, ČSAV, Prague (in Czech).

Christensen, R.M. (1982). *The Theory of Viscoelasticity*, Academic Press, New York.

Eringen, A.C. (1967). *Mechanics of Continua*, J. Wiley, New York.

Eringen, A.C. and E.S. Suhubi (1974). *Elastodynamics. Finite Motions*, Academic Press, New York.

Haupt, P. (1993). Foundation of continuum mechanics. In: *Continuum Mechanics in Environmental Sciences and Geophysics*, ed. C. Hutter, Springer-Verlag, Berlin, 1-77.

Malvern, L.E. (1969). *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Englewood Cliffs.

Mase, G.E. and Mase, G.T. (1992). *Continuum Mechanics for Engineers*, CRC Press, Boca Raton.

Nečas, J. and I. Hlaváček (1981). *Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction*, Elsevier, Amsterdam.

Truesdell, C. (1972). *A First Course in Rational Continuum Mechanics*, J. Hopkins University, Baltimore.

Truesdell, C. and W. Noll (1965). *The Nonlinear Field Theories of Mechanics*, Springer-Verlag, Berlin.