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REGRESSION MODELS WITH A WEAK NONLINEARITY

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Regression models with a weak nonlinearity[1]

Introduction

In a series of statistical problems which are to be solved using the regression models the nonlinear models occur, where the mean value of the measurement, the constraints binding the unknown parameters and hypotheses concerning them are not linear.

In the practice there exist two different approaches to solving the nonlinear problems. Either the model is linearized and the least square method for the linear model is used or the nonlinear least square method is applied for determining the estimator of the unknown parameters.

In the former case if the linearization procedure is justified the statistical considerations may be performed using well elaborated methods that are at our disposal for estimating the unknown mean value and covariance matrix parameters, for determining the confidence regions, for the power function of the test, etc. These procedures are relatively simple and moreover they are at our disposal in standard package of programs.

The latter approach is more adequate. However series of problems that are in practice to be solved are not worked out yet or their solution is more difficult than in the linear case. If it is possible to apply the former approach without damaging the quality of the solution, it will be preferred in applications.

The linearization of nonlinear models conseals certain risks. In order to eliminate them it is necessary to know some criteria concerned the linearization regions that have to be satisfied in order not to deteriorate the statistical procedures applied.

The aim of the contribution is to elaborate a system of these criteria for various statistical tasks that occur in the nonlinear regression models.

Consider regression model $Y \sim_n [f(\beta), \Sigma]$, $\beta \in \mathcal{V} \subset R^k$, where $f : \mathcal{V} \to R^n$ is a known function possessing continuous derivatives including the second

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ones, Σ a known positive (p.d.) matrix, $\mathcal{V} \subset R^k$ a parametric space and R^k a k-dimensional Euklidean space.

Generally, this model is said to be a model with a weak nonlinearity if in the Taylor series of the function $f(\cdot)$ at the approximate point β_0 the terms of the second and higher degrees may be neglected. The shapes of the linearization region, i.e. of those values of the difference $\delta\beta = \beta - \beta_0$ that are from various points of view admissible for the linearization of the model are in particular defined by formulas (7), (10), (12), (14), (15), (19), (20), (24), (26), etc.

In the following the models

- without constraints, when $\mathcal{V} = R^k$,
- with constraints of the type I, when $\mathcal{V} = \mathcal{V}_I = \{\beta : g(\beta) = 0\}$, where $g(\cdot)$ is a known q-dimensional vector function with continuous second derivatives and
- with constraints of the type II, when $\mathcal{V} = \mathcal{V}_{II} = \left\{ \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} : h(\beta_1, \beta_2) = 0 \right\}$, where $h(\cdot)$ is a known q-dimensional function with continuous second derivatives

are considered. For further structures see [3].

1 Curvature in regression models

In this section the model

$$Y \sim_n [f(\beta), \Sigma], \; \beta \in R^k, \tag{1}$$

where $f(\cdot)$, $\beta \in R^k$, is a known function with continuous derivatives, $\Sigma = Var(Y)$ is a known positive definite matrix, is considered.

Definition 1.1 The Bates and Watts measures of curvature of the model (1) at the point β are the intrinsic curvature

$$K^{(int)}(\beta) = \sup \left\{ \frac{||M_F^{\Sigma^{-1}} \kappa_s||_{\Sigma^{-1}}}{||Fs||^2_{\Sigma^{-1}}} : s \in R^k \right\}, \tag{2}$$

where

$$F = \partial f(u)/\partial u'|_{u=\beta}, \; \kappa_s = (s'F_1 s, \dots s'F_n s)' \; F_i = \partial^2 f_i(u)/\partial u \partial u'|_{u=\beta},$$

$i = 1, \ldots, n$, $M_F^{\Sigma^{-1}} = I - P_F^{\Sigma^{-1}}$ and $P_F^{\Sigma^{-1}}$ is the projector on $\mathcal{M}(F)$ in the norm Σ^{-1} and the parametric curvature

$$K^{(par)}(\beta) = \sup\left\{\frac{||P_F^{\Sigma^{-1}}\kappa_s||_{\Sigma^{-1}}}{||Fs||^2_{\Sigma^{-1}}} : s \in R^k\right\}, \tag{3}$$

see [1] and [2]. In model (1) the most natural estimator of the unknown vector β is a solution $\hat{\beta}(Y)$ of the equation

$$\frac{\partial}{\partial u}[Y - f(u)]'\Sigma^{-1}[Y - f(u)] = 0 \tag{4}$$

called a nonlinear mean square estimator. In order to obtain just one solution of the problem, the realization y of the observation vector Y must lay sufficiently near to the solution locus (the mean value manifold) $\{f(\beta) : \beta \in R^k\}$. A more precise formulation contains the next lemma.

Lemma 1.2 Let the matrix $F = \partial f(u)/\partial u'|_{u=\hat{\beta}}$, where $\hat{\beta}$ is any solution of the equation (4), be of full rank in columns. Then the inequality

$$||y - f(\hat{\beta})||_{\Sigma^{-1}} < \frac{1}{K^{(int)}(\hat{\beta})}$$

implies that the matrix

$$H = \frac{\partial^2}{\partial u \partial u'}\{[y - f(u)]'\Sigma^{-1}[y - f(u)]\}\Big|_{u=\hat{\beta}}$$

is positive definite.

Proof.

$$\begin{aligned} &\frac{\partial^2}{\partial u_i \partial u'}\{[y - f(u)]'\Sigma^{-1}[y - f(u)]\} = \\ &= \ -2y'\Sigma^{-1}\frac{\partial^2 f(u)}{\partial u_i \partial u'} + 2\frac{\partial f'(u)}{\partial u_i}\Sigma^{-1}\frac{\partial f(u)}{\partial u'} + 2f'(u)\Sigma^{-1}\frac{\partial^2 f(u)}{\partial u_i \partial u'}. \end{aligned}$$

According to the definition the matrix H is positive definite iff for any $x \neq 0,\ x \in R^k$,

$$-2y'\Sigma^{-1}\sum_{i=1}^k x_i \frac{\partial^2 f(u)}{\partial u_i \partial u'}x + 2\sum_{i=1}^k x_i \frac{\partial f'(u)}{\partial u_i}\Sigma^{-1}\frac{\partial f(u)}{\partial u'}x + 2f'(u)\Sigma^{-1}\sum_{i=1}^k x_i \frac{\partial^2 f(u)}{\partial u_i \partial u'}x > 0,$$

i.e. iff

$$x'F'\Sigma^{-1}Fx > (y - f)'\Sigma^{-1}\kappa_x;$$

here $F = \partial f(u)\partial u'|_{u=\hat{\beta}}$ and $f = f(\hat{\beta})$.

Further, since (4) can be written in the form $F'\Sigma^{-1}(y-f)=0$ (Σ^{-1}-orthogonality of $F'\Sigma^{-1}$ and $y-f$), we see that

$$y-f \in \mathcal{M}\left(M_F^{\Sigma^{-1}}\right) \Rightarrow y-f = M_F^{\Sigma^{-1}}(y-f).$$

Thus $(y-f)'\Sigma^{-1}\kappa_x = (y-f)'(M_F^{\Sigma^{-1}})'\Sigma^{-1}\kappa_x$. By applying now the Schwarz inequality, we obtain

$$\left|(y-f)'(M_F^{\Sigma^{-1}})'\Sigma^{-1}\kappa_x\right| \leq ||y-f||_{\Sigma^{-1}}||M_F^{\Sigma^{-1}}\kappa_x||_{\Sigma^{-1}}.$$

Using the definition of the intrinsic curvature given by (2) we get

$$\frac{|(y-f)'\Sigma^{-1}\kappa_x|}{||Fx||^2_{\Sigma^{-1}}} \leq ||y-f||_{\Sigma^{-1}}K^{(int)}(\hat{\beta})$$

and if

$$||y-f||_{\Sigma^{-1}} < \frac{1}{K^{(int)}(\hat{\beta})},$$

then

$$||Fx||^2_{\Sigma^{-1}} = x'F'\Sigma^{-1}Fx > |(y-f)'\Sigma^{-1}\kappa_x| \geq (y-f)'\Sigma^{-1}\kappa_x,$$

thus the matrix H is positive definite. ♠

Now it is clear that the accuracy of the measurement characterized by the covariance matrix Σ of the observation vector has to be so high that the distance between the realized observation vector and the mean value manifold be with a probability near to 1 smaller than the inverse value of its intrinsic curvature $1/K^{(int)}$, i.e. than the smallest curvature radius of the osculation circle of the normal section of the mean value manifold at the point $f=f(\hat{\beta})$.

To obtain the estimator $\hat{\beta}(Y)$ of the parameter β directly as a solution of the equation (4) and an investigation of its statistical properties may become difficult (cf. e.g. [7]). The problems connected with this approach may be avoided if we know "with a sufficiently high accuracy" the approximate value β_0 of the unknown parameter β, whose actual value is β^*, i.e. if we know a value β lying in a "sufficiently small neighbourhood of β^*." Under this condition the function $f(\beta)$ in the model (1) can be developed into the Taylor series

$$f(\beta) = f(\beta_0) + F(\beta_0)\delta\beta + \frac{1}{2}\kappa_{\delta\beta} + \ldots,$$

where

$$F(\beta_0) = \partial f(\beta)/\partial\beta'|_{\beta=\beta_0},\ \kappa_{\delta\beta} = (\delta\beta' F_1\delta\beta, \ldots \delta\beta' F_n\delta\beta)',$$
$$F_i = \partial^2 f_i(\beta)/\partial\beta\partial\beta'|_{\beta=\beta_0},\ i=1,\ldots,n$$

and $\delta\beta = \beta - \beta_0$. If we neglect the terms of the degree higher than the first order, instead of the general model (1) we get the linearized model

$$Y \sim_n (f_0 + F\delta\beta, \Sigma); \tag{5}$$

here the abbreviated notation $f_0 = f(\beta_0)$ and $F = \partial f(\beta)/\partial\beta|_{\beta=\beta_0}$ was used. The linearized model (5) can be handled easily.

Nevertheless within this approach the problem has to be solved when the assumption "β_0 is lying in a sufficiently small neighbourhood of β^*" is fulfilled. The neglected terms of the Taylor development of the function $f(\cdot)$, $\beta \in R^k$, can influence the bias of the estimator $\hat{\beta}$ of β, its variance, confidence region, the estimator of the unit dispersion etc.

In the first step the problem must be solved whether the realization y of the observation vector Y is or is not consistent with the linearized model.

In order to solve the problem we test within a quadratic model $Y - f_0 \sim_n (F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \Sigma)$ the null hypothesis $H_0 : \delta = 0$ against the alternative $H_a : \delta \neq 0$ using the test statistic (6) and search for the region of those values $\delta\beta$ that correspond the given maximum admissible increase ε of the risk α of the test (it determines the maximum admissible value $\delta_{\max}$ of the noncentrality parameter of the test statistic used).

Lemma 1.3 Let $Y - f_0 \sim N_n(F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \Sigma)$. Then

1. the test statistic
$$(Y - f_0)'\Sigma^{-1}M_F^{\Sigma^{-1}}(Y - f_0) \tag{6}$$
possesses a noncentral chi-squared probability distribution with $n - k$ degrees of freedom and with the noncentrality parameter
$$\delta = \frac{1}{4}\kappa'_{\delta\beta}\Sigma^{-1}M_F^{\Sigma^{-1}}\kappa_{\delta\beta};$$

2. if
$$\delta\beta' C \delta\beta < \frac{2\sqrt{\delta_{\max}}}{K^{(int)}(\beta_0)},$$
where $C = F'\Sigma^{-1}F$ and $\delta_{\max}$ for the given ε is given by the solution of the equation
$$P\{\chi^2_{n-k}(\delta_{\max}) \leq \chi^2_{n-k}(0; 1-\alpha)\} = 1 - (\alpha + \varepsilon)$$
($\chi^2_{n-k}(0; 1-\alpha)$ being the $1-\alpha$ quantile of the central chi-squared probability distribution with $n-k$ degrees of freedom), then
$$P\{(Y - f_0)'\Sigma^{-1}M_F^{\Sigma^{-1}}(Y - f_0) \leq \chi^2_{n-k}(0; 1-\alpha)\} \geq 1 - \alpha - \varepsilon.$$

Proof. 1 is obvious.

2. Since

$$\delta = \frac{1}{4}\kappa'_{\delta\beta}\Sigma^{-1}M_F^{\Sigma^{-1}}\kappa_{\delta\beta}$$

and

$$\frac{\frac{1}{4}\kappa_{\delta\beta}\Sigma^{-1}M_F^{\Sigma^{-1}}\kappa_{\delta\beta}}{\frac{1}{4}||F\delta\beta||^4_{\Sigma^{-1}}} \leq [K^{(int)}(\beta_0)]^2,$$

we have $\delta < \delta_{\max}$, i.e. if

$$\frac{1}{4}||F\delta\beta||^4_{\Sigma^{-1}}[K^{(int)}(\beta_0)]^2 < \delta_{\max},$$

that occurs iff

$$\delta\beta' C\delta\beta < \frac{2\sqrt{\delta_{\max}}}{K^{(int)}(\beta_0)},$$

then

$$P\{(Y - f_0)'\Sigma^{-1}M_F^{\Sigma^{-1}}(Y - f_0) = \chi^2_{n-k}(\delta) \leq \chi^2_{n-k}(0; 1-\alpha)\} \geq 1 - \alpha - \varepsilon.$$

Thus within a quadratic approximation of model (1) (under the condition of normality of the observation vector) for a negligible ε determining $\delta_{\max}$ the quadratic model can be considered to be linear on the set

$$\left\{\beta_0 + \delta\beta : \delta\beta' C\delta\beta \leq \frac{2\sqrt{\delta_{\max}}}{K^{(int)}(\beta_0)}\right\} \tag{7}$$

in the sence of consistency of the data. ♠

For another approach to testing fit for linear models respecting only those components of the residual vector that are sensitive to the alternative hypothesis see [7, pp. 49–51].

Remark 1.4 The general form of the quantities $K^{(int)}(\beta_0)$, $K^{(par)}(\beta_0)$ and further measures of curvature given in the following sections is

$$C(\beta_0) = \sup\left\{\frac{\sqrt{\alpha'_{\delta u}S\alpha_{\delta u}}}{\delta u' R\delta u} : \delta u \in R^k\right\},$$

where

$$\alpha_{\delta u} = (\delta u' A_1 \delta u, \dots, \delta u' A_r \delta u)$$

(see, e.g.(2) and (3)), where $A_1, \dots, A_r$ are $k \times k$ symmetric matrices, S is an $r \times r$ positive semidefinite matrix and R a $k \times k$ positive definite matrix.

There exists a simple iteration procedure for calculating their values.

1st step: We choose an arbitrary vector $\delta u_1 \in R^k$ such that $\delta u_1'\delta u_1 = 1$.

2nd step:

- We determine the vector δs

$$\delta s = R^{-1}(A_1\delta u_1, \dots, A_r\delta u_1)S \begin{pmatrix} \delta u_1' A_1 \delta u_1 \\ \vdots \\ \delta u_1' A_r \delta u_1 \end{pmatrix}$$

and using it

- we determine the vector δu_2

$$\delta u_2 = \delta s / \sqrt{\delta s' \delta s}.$$

3rd step:

- If $\delta u_2' \delta u_1 \geq 1 - \varepsilon$, where ε is a sufficiently small positive number, the iteration procedure is finished and $C(\beta_0)$ is determined as

$$C(\beta_0) = \left\{ \frac{\sqrt{\alpha_{\delta u_2}' S \alpha_{\delta u_2}}}{\delta u_2' R \delta u_2} \right\}.$$

- If $\delta u_2' \delta u_1 < 1 - \varepsilon$, we return to the first step with the vector δu_2 used instead of the vector δu_1.

2 Nonlinearity in the bias of estimators

2.1 The model without constraints

Let us consider model (1) in its quadratic approximation

$$Y - f_0 \sim_n \left(F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \Sigma \right), \ \delta\beta \in R^k, \tag{8}$$

where $\delta\beta = \beta - \beta_0$, β_0 is an approximate value of β, $f_0 = f(\beta_0)$, $F = \partial f(\beta)/\partial\beta'|_{\beta=\beta_0}$, $\kappa_{\delta\beta} = (\delta\beta' F_1 \delta\beta, \dots, \delta\beta' F_n \delta\beta)'$, $F_i = \partial^2 f_i(\beta)/\partial\beta\partial\beta'|_{\beta=\beta_0}$, $i = 1, \dots, n$.

Lemma 2.1.1 If $r(F) = k < n$ and Σ is positive definite, then the bias of the estimator $\delta\hat{\beta}(Y, \beta_0) = C^{-1}F'\Sigma^{-1}(Y - f_0)$ is

$$b = \frac{1}{2} C^{-1} F' \Sigma^{-1} \kappa_{\delta\beta}, \tag{9}$$

where $C = F'\Sigma^{-1}F$.

Proof. (9) follows directly from the definition of a bias. ♠

Corollary 2.1.2 If

$$\delta\beta' C \delta\beta \leq \frac{2c_b\sqrt{\chi_k^2(0,1-\alpha)}}{K^{(par)}(\beta_0)}, \tag{10}$$

then

$$b'Cb \leq c_b^2 \chi_k^2(0;1-\alpha),$$

where c_b is a positive constant chosen by user (for its meaning see Remark 2.1.8).

Proof. With respect to (9)

$$b'Cb = \frac{1}{4}\kappa'_{\delta\beta}\Sigma^{-1}F'C^{-1}CC^{-1}F\Sigma^{-1}\kappa_{\delta\beta} = \frac{1}{4}\kappa'_{\delta\beta}\Sigma^{-1}P_F^{\Sigma^{-1}}\kappa_{\delta\beta} = \frac{1}{4}||P_F^{\Sigma^{-1}}\kappa_{\delta\beta}||^2_{\Sigma^{-1}}.$$

With respect to the definition of the parametric curvature (see(3)) we get

$$\frac{b'Cb}{\frac{1}{4}||F\delta\beta||^4_{\Sigma^{-1}}} \leq [K^{(par)}(\beta_0)]^2,$$

what directly implies that

$$b'Cb \leq \frac{1}{4}[K^{(par)}(\beta_0)]^2(\delta\beta' C\delta\beta)^2$$

and if we take into account the condition (10), we get immediately

$$b'Cb \leq \frac{1}{4}[K^{(par)}(\beta_0)]^2\frac{4c_b^2\chi_k^2(0;1-\alpha)}{[K^{(par)}(\beta_0)]^2} = c_b^2\chi_k^2(0;1-\alpha).$$

♠

Lemma 2.1.3

$$b'Cb \leq c_b^2\chi_k^2(0;1-\alpha)$$

if and only if

$$\forall\{h \in R^k\}|h'b| \leq c_b\sqrt{\chi_k^2(0;1-\alpha)}\sqrt{h'C^{-1}h}.$$

Proof. The assertion is a direct consequence of the Scheffé theorem [8, p.69]. ♠

Corollary 2.1.4 If

$$\delta\beta' C \delta\beta \leq \frac{2c_b\sqrt{\chi_k^2(0,1-\alpha)}}{K^{(par)}(\beta_0)},$$

cf. condition (10), then

$$\forall\{h \in R^k\}|h'b| \le c_b\sqrt{\chi_k^2(0;1-\alpha)}\sigma_h,\ \sigma_h = \sqrt{h'C^{-1}h}. \tag{11}$$

Thus the joint linearization region for all linear functions of β is

$$\left\{\delta\beta : \delta\beta' C \delta\beta \le \frac{2c_b\sqrt{\chi_k^2(0;1-\alpha)}}{K^{(par)}(\beta_0)}\right\}, \tag{12}$$

i.e. (12) is a region of $\delta\beta$ on which any linear function $h'b$ of the bias of the estimator $\delta\hat{\beta}(Y,\beta_0) = C^{-1}F'\Sigma^{-1}(Y - f_0)$ does not exceed the value $c_b\sqrt{\chi_k^2(0;1-\alpha)}\sigma_h$.

Proof. It is a direct consequence of Corollary 2.1.2 and Lemma 2.1.3. ♠

The region (12) is a joint linearization region holding for every linear function of the parameter β. In the following we focus our attention to a concrete fixed linear function $h'\beta$.

Lemma 2.1.5 Consider a linear function $h(\beta) = h'\beta, \beta \in R^k$, where $h \in R^k$ is a fixed vector. If

$$\delta\beta'\overline{A}_h\delta\beta \le \varepsilon_{b,h}\sigma_h, \tag{13}$$

then also

$$|h'b| \le \varepsilon_{b,h}\sigma_h. \tag{14}$$

Here $A_h = \sum_{i=1}^n \frac{1}{2}L_{h,i}H_i = \sum_{i=1}^k \lambda_i f_i f_i'$ (spectral decomposition of the matrix A_h), $\overline{A}_h = \sum_{i=1}^k |\lambda_i| f_i f_i'$, $L_{h,i} = \{h'C^{-1}F'\Sigma^{-1}\}_i$, $i = 1,\dots,n$ and $\varepsilon_{b,h}$ is a positive constant depending on h, chosen by the user.

Proof. Within model (8)

$$h'b = \frac{1}{2}h'C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta}.$$

If we denote $h'C^{-1}F'\Sigma^{-1} = L_h' = (L_{h,1},\dots,L_{h,n})$ and $A_h = \sum_{i=1}^n \frac{1}{2}L_{h,i}H_i$, then $h'b = \delta\beta' A_h\delta\beta$. Further let the spectral decomposition of the matrix A_h be $A_h = \sum_{i=1}^k \lambda_i f_i f_i'$ and let $\overline{A}_h = \sum_{i=1}^k |\lambda_i| f_i f_i'$. Then $|h'b| \le \varepsilon_{b,h}\sigma_h$ if $\delta\beta'\overline{A}_h\delta\beta$ (conceive that $h'b = \delta\beta' A_h\delta\beta \le |h'b| \le \delta\beta'\overline{A}_h\delta\beta$). ♠

Corollary 2.1.6 The linearization region for a fixed linear function of the unknown parameter β based on the inequality (13) is

$$\{\delta\beta : \delta\beta'\overline{A}_h\delta\beta \le \varepsilon_{b,h}\sigma_h\}. \tag{15}$$

Remark 2.1.7 It is reasonable to compare the linearization regions (12) and (15). It can happen that (12) holding for all linear functions $h'\beta$ is for some important function $h(\beta) = h'\beta$, $\beta \in R^k$, too small while (15) corresponding a concrete function with a fixed $h \in R^k$ is sufficiently larger. In such a case the region (15) can be advantageously used.

Remark 2.1.8 Lemma 2.1.5 can be easily applied for interpreting the meaning of constant c_b from Corollary 2.1.2 (cf. relation (10)).

If we compare inequality (11) from Corollary 2.1.4 holding for each $h \in R^k$ with the inequality (14) from Lemma 2.1.5 holding for a concrete $h \in R^k$, we get

$$c_b\sqrt{\chi_k^2(0; 1-\alpha)} = \varepsilon_b,$$

where c_b is a chosen positive constant independent of h.

2.2 The model with constraints of the type I

Let us consider model (1) in its quadratic approximation

$$Y - f_0 \sim_n \left(F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \Sigma\right)$$

under the condition $\delta\beta \in \mathcal{V} = \{\beta : g(\beta) = 0\}$, $g(\cdot)$ is a known q-dimensional vector function with continuous second derivatives.

The following notation will be used:

$$g_0 = g(\beta_0),\ \partial g(\beta)/\partial\beta'\Big|_{\beta=\beta_0} = G,\ \gamma_{\delta\beta} = \begin{pmatrix} \delta\beta' G_1 \delta\beta \\ \vdots \\ \delta\beta' G_q \delta\beta \end{pmatrix},\ G_i = \frac{\partial^2 g_i(\beta)}{\partial\beta\partial\beta'}\Big|_{\beta=\beta_0},$$

$i = 1, \ldots, q$.

Lemma 2.2.1 Let $Y - f_0 \sim_n (F\delta\beta, \Sigma)$, $\delta\beta \in \{\delta\beta : g_0 + G\delta\beta = 0\}, r(F) = k < n$, $R(G) = q < k$ and Σ be positive definite. Then the BLUE of $\delta\beta$ is

$$\begin{aligned} &\delta\hat{\hat{\beta}}(Y, \beta_0) = \\ &= [I - C^{-1}G'(GC^{-1}G')^{-1}G]C^{-1}F'\Sigma^{-1}(Y - f_0) - C^{-1}G'(GC^{-1}G')^{-1}g_0 = \\ &= M^C_{C^{-1}G'}C^{-1}F'\Sigma^{-1}(Y - f_0) - C^{-1}G'(GC^{-1}G')^{-1}g_0. \qquad (16) \end{aligned}$$

In (16) $C^{-1}F'\Sigma^{-1}(Y - f_0) = \delta\hat{\beta}(Y, \beta_0)$ is the BLUE in model $Y - f_0 \sim_n (F\delta\beta, \Sigma)$, $\delta\beta \in R^k$.

Corollary 2.2.2 If $Y - f_0 \sim_n (F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \Sigma)$ and $\delta\beta \in \{\delta\beta : G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0\}$, then the bias of the BLUE of the form (16) is

$$b = \frac{1}{2}M^C_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta} + \frac{1}{2}C^{-1}G'(GC^{-1}G')^{-1}\gamma_{\delta\beta}. \qquad (17)$$

Proof. It is a direct consequence of the definition of a bias of an estimator and of the fact that

$$
\begin{aligned}
& E(\hat{\hat{\beta}}|\beta) = \\
&= M^{C}_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\left(F\delta\beta + \frac{1}{2}\kappa\delta\beta\right) + C^{-1}G'(GC^{-1}G')^{-1}\left(G\delta\beta + \frac{1}{2}\gamma_{\delta\beta}\right) = \\
&= \delta\beta + \frac{1}{2}M^{C}_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta} + \frac{1}{2}C^{-1}G'(GC^{-1}G')^{-1}\gamma_{\delta\beta}.
\end{aligned}
$$

♠

Let K_G be a $k \times (k-q)$ matrix with the property $\mathcal{K}er(G) = \mathcal{M}(K_G)$.

Theorem 2.2.3 Let

$$
K_I^{(constr)}(\beta_0) = \sup\left\{\frac{\sqrt{\gamma'_{K_G\delta s}(GC^{-1}G')^{-1}\gamma_{K_G\delta s}}}{\delta s' K'_G C K_G \delta s} : \delta s \in R^{k-q}\right\}. \tag{18}
$$

Then

1.
$$
\delta s' K'_G C K_G \delta s \le \frac{2\varepsilon}{K_I^{(constr)}(\beta_0)} \tag{19}
$$
implies
$$
\forall\left\{h \in \mathcal{M}\left[\left(P^{C}_{C^{-1}G'}\right)'\right]\right\} |h'b| \le \varepsilon\sigma_h
$$
and

2.
$$
\delta\beta' C \delta\beta \le \frac{2\varepsilon}{K^{(par)}(\beta_0)} \tag{20}
$$
implies
$$
\forall\{h \in \mathcal{M}[(M^{C}_{C^{-1}G'})']\}|h'b| \le \varepsilon\sigma_h;
$$
here b is given by (17), $\sigma_h = \sqrt{h'C^{-1}h}$ and ε is a given positive constant.

Proof. 1. Let $h \in \mathcal{M}[(P^{C}_{C^{-1}G'})']$ (i.e. there exists u such that $h' = u'P^{C}_{C^{-1}G'}$) then

$$
\begin{aligned}
|h'b| &= \frac{1}{2}\left|u'P^{C}_{C^{-1}G'}[M^{C}_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta} + C^{-1}G'(GC^{-1}G')^{-1}\gamma_{\delta\beta}]\right| = \\
&= \frac{1}{2}|h'C^{-1}G'(GC^{-1}G')^{-1}\gamma_{\delta\beta}|
\end{aligned}
$$

and by applying now the Schwarz inequality we get

$$\begin{aligned}
&\left|\frac{1}{2}h'C^{-1}G'(GC^{-1}G')^{-1}\gamma_{\delta\beta}\right| \leq \\
&\leq \sqrt{h'C^{-1}h}\sqrt{\frac{1}{4}\gamma'_{\delta\beta}(GC^{-1}G')^{-1}GC^{-1}CC^{-1}G'(GC^{-1}G')^{-1}\gamma_{\delta\beta}} \\
&= \sigma_h\frac{1}{2}\sqrt{\gamma'_{\delta\beta}(GC^{-1}G')^{-1}\gamma_{\delta\beta}} \\
&\leq \sigma_h K_I^{(constr)}(\beta_0)\frac{1}{2}\delta\beta'[C-G'(GC^{-1}G')^{-1}G]\delta\beta
\end{aligned}$$

and now if we apply the assumption (19)

$$\delta s' K_G' C K_G \delta s = \delta\beta'[C-G'(GC^{-1}G')^{-1}G]\delta\beta \leq \frac{2\varepsilon}{K_I^{(constr)}(\beta_0)},$$

we get assertion 1.

Ad 2. Let $h \in \mathcal{M}[(M^C_{C^{-1}G'})']$, then

$$\begin{aligned}
|h'b| &= \frac{1}{2}\left|u'M^C_{C^{-1}G'}[M^C_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta}+C^{-1}G'(GC^{-1}G')^{-1}\gamma_{\delta\beta}]\right| \\
&= \frac{1}{2}|h'C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta}|;
\end{aligned}$$

by applying the Schwarz inequality we get

$$\begin{aligned}
\left|\frac{1}{2}h'C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta}\right| &\leq \sqrt{h'C^{-1}h}\sqrt{\frac{1}{4}\kappa'_{\delta\beta}\Sigma^{-1}F'C^{-1}CC^{-1}F\Sigma^{-1}\kappa_{\delta\beta}} \\
&= \sigma_h\frac{1}{2}\sqrt{\kappa'_{\delta\beta}\Sigma^{-1}P_F^{\Sigma^{-1}}\kappa_{\delta\beta}} \leq \\
&\leq \sigma_h K^{(par)}(\beta_0)\frac{1}{2}\delta\beta' C\delta\beta.
\end{aligned}$$

If we apply now the assumption

$$\delta\beta' C\delta\beta \leq \frac{2\varepsilon}{K^{(par)}(\beta_0)},$$

we get the assertion 2. ♠

Remark 2.2.4 The model $Y - f_0 \sim_n \left(F\delta\beta + \frac{1}{2}\kappa_{\delta\beta},\ \Sigma\right)$ with constraints $G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0$ is identical until the second order terms with the model

$$Y - f_0 \sim_n \left(FK_G\delta u + \frac{1}{2}(\kappa_{K_G\delta u} - FG^-_{m(C)}\gamma_{K_G\delta u}), \Sigma\right),\ \delta u \in R^{k-q}, \quad (21)$$

where K_G is a kernel matrix of the matrix G (i.e. a $k \times (k-q)$ matrix with the rank $r(K_G) = k - q$ possesing the property $GK_G = 0$), δu is a new $(k-q)$ dimensional parameter arising from the linearized condition $G\delta\beta = 0$ ($g_0 = g(\beta_0) \equiv 0$ - the condition must be satisfied by the approximate value of the unknown parameter) whose nontrivial solution is of the form $\delta\beta = K_G\delta u$, $\delta u \in R^{k-q}$ and $G^-_{m(C)}$ is the minimum C- norm g-inverze of the matrix G (cf. [4]).

Proof. It suffices to realize that $\delta\beta = K_G\delta u - \frac{1}{2}G^-_{m(C)}\gamma_{K_G\delta u}$ fulfils the quadratic condition $G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0$ inclusive the quadratic terms. ♠

Definition 2.2.5 The Bates and Watts measures of curvature of the model (21) at the point β_0 are the I-intrinsic curvature

$$\begin{aligned} K_I^{(int)}(\beta_0) &= \\ &= \sup\left\{\frac{\sqrt{\boxed{1}'\Sigma^{-1}M_{FK_G}^{\Sigma^{-1}}\boxed{1}}}{\delta u' K_G' C K_G \delta u} : \delta u \in R^{k-q}\right\} \\ &= \sup\left\{\frac{\sqrt{\boxed{1}'\Sigma^{-1}\boxed{1} - \kappa_{K_G\delta u}'\Sigma^{-1}P_{FK_G}^{\Sigma^{-1}}\kappa_{K_G\delta u}}}{\delta u' K_G' C K_G \delta u} : \delta u \in R^{k-q}\right\}, \end{aligned} \tag{22}$$

and the I-parametric curvature

$$\begin{aligned} K_I^{(par)}(\beta_0) &= \\ &= \sup\left\{\frac{\sqrt{\boxed{1}'\Sigma^{-1}P_{FK_G}^{\Sigma^{-1}}\boxed{1}}}{\delta u' K_G' C K_G \delta u} : \delta u \in R^{k-q}\right\} \\ &= \sup\left\{\frac{\sqrt{\kappa_{K_G\delta u}'\Sigma^{-1}P_{FK_G}^{\Sigma^{-1}}\kappa_{K_G\delta u}}}{\delta u' K_G' C K_G \delta u} : \delta u \in R^{k-q}\right\}, \end{aligned} \tag{23}$$

where $\boxed{1} = \kappa_{K_G\delta u} - FG^-_{m(C)}\gamma_{K_G\delta u}$ (since $P_{FK_G}^{\Sigma^{-1}}FG^-_{m(C)} = 0$ for $G^-_{m(C)} = C^{-1}C(GC^{-1}G')^{-1}$).

Theorem 2.2.6 Let

$$K_I^{(par)}(\beta_0) = \sup\left\{\frac{\sqrt{\kappa_{K_G\delta u}'\Sigma^{-1}P_{FK_G}^{\Sigma^{-1}}\kappa_{K_G\delta u}}}{\delta u' K_G' C K_G \delta u} : \delta u \in R^{k-q}\right\}.$$

Then

$$\delta u' K_G' C K_G \delta u \leq \frac{2\varepsilon}{K_I^{(par)}(\beta_0)} \tag{24}$$

implies

$$\forall\{h \in \mathcal{M}[(M_{C^{-1}G'}^{C})']\} |h'b| \leq \varepsilon\sigma_h.$$

Proof. Let $h \in \mathcal{M}[(M^C_{C^{-1}G'})']$, then

$$|h'b| = \frac{1}{2}|h'C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta}| = \frac{1}{2}\left|h'M^C_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta}\right|;$$

By applying the Schwarz inequality we get

$$\begin{aligned}
&\left|\frac{1}{2}h'M^C_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta}\right| \leq \\
\leq\ & \sqrt{h'C^{-1}h}\sqrt{\frac{1}{4}\kappa'_{\delta\beta}\Sigma^{-1}C^{-1}(M^C_{C^{-1}G'})'CM^C_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta}} \\
=\ & \sigma_h\frac{1}{2}\sqrt{\kappa'_{K_G\delta u}\Sigma^{-1}P^{\Sigma^{-1}}_{FK_G}\kappa_{K_G\delta u}} \\
\leq\ & \sigma_h \sup\left\{\frac{\sqrt{\kappa'_{K_G\delta u}\Sigma^{-1}P^{\Sigma^{-1}}_{FK_G}\kappa_{K_G\delta u}}}{\delta u'K'_GCK_G\delta u} : \delta u \in R^{k-q}\right\}\frac{1}{2}\delta u'K'_GCK_G\delta u \\
=\ & \frac{1}{2}\sigma_h K^{(par)}_I(\beta_0)\delta u'K'_GCK_G\delta u
\end{aligned}$$

and for the assumed inequality (24) implies the assertion of the theorem.

The facts that

$$\begin{aligned}
(M^C_{C^{-1}G'})'CM^C_{C^{-1}G'} &= CM^C_{C^{-1}G'}, \\
M^C_{C^{-1}G'}C^{-1} &= (M_{G'}CM_{G'})^+ = M_{G'}(M_{G'}CM_{G'})^+M_{G'}, \\
FM_{G'}(M_{G'}CM_{G'})^+M_{G'}F'\Sigma^{-1} &= P^{\Sigma^{-1}}_{FM_{G'}} = P^{\Sigma^{-1}}_{FK_G}
\end{aligned}$$

were taken into account. ♠

Definition 2.2.7 The I-parametric measure of nonlinearity of the model $Y - f_0 \sim_n \left(F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \Sigma\right)$ with the constraints $G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0$ with respect to the parameter $\delta\beta$ at the point β_0 is defined as

$$\begin{aligned}
&C^{(par)}_{I,\delta\beta}(\beta_0) = \\
&= \sup\left\{\frac{\sqrt{\kappa'_{K_G\delta u}\Sigma^{-1}P^{\Sigma^{-1}}_{FK_G}\kappa_{K_G\delta u} + \gamma'_{K_G\delta u}(GC^{-1}G')^{-1}\gamma_{K_G\delta u}}}{\delta u'K'_GF'\Sigma^{-1}FK_G\delta u} : \delta u \in R^{k-q}\right\}.
\end{aligned} \tag{25}$$

Theorem 2.2.8 If

$$\delta u'K'_GCK_G\delta u \leq \frac{2\varepsilon}{C^{(par)}_{I,\delta\beta}(\beta_0)}, \tag{26}$$

then

$$\forall\{h \in R^k\}|h'b| \leq \varepsilon\sigma_k,$$

where $\varepsilon > 0$ is a positive chosen number.

Proof. According to (17)

$$|h'b| = \frac{1}{2}\left|h'\left(M^C_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta} + C^{-1}G'(GC^{-1}G')^{-1}\gamma_{\delta\beta}\right)\right|.$$

Applying the Schwarz inequality we get

$$\begin{aligned}|h'b| &\leq \\ &\leq \sqrt{h'C^{-1}h}\frac{1}{2}\left([\kappa'_{\delta\beta}\Sigma^{-1}FC^{-1}(M^C_{C^{-1}G'})' + \gamma'_{\delta\beta}(GC^{-1}G')^{-1}GC^{-1}]C \times \right. \\ &\quad \left. \times[M^C_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta} + C^{-1}G'(GC^{-1}G')^{-1}\gamma_{\delta\beta}]\right)^{1/2} \\ &= \sigma_h\frac{1}{2}\sqrt{\kappa'_{K_G\delta u}\Sigma^{-1}P^{\Sigma^{-1}}_{FK_G}\kappa_{K_G\delta u} + \gamma'_{K_G\delta u}(GC^{-1}G')^{-1}\gamma_{K_G\delta u}};\end{aligned}$$

taking into account Definition 2.2.7 and the assumption (26), we obtain the assertion of the theorem.

Here in addition to the facts mentioned in the previous theorem the equality $GM^C_{C^{-1}G'} = 0$ was taken into account. ♠

Remark 2.2.9 If $h \in \mathcal{M}\left[\left(M^C_{C^{-1}G'}\right)'\right]$, then $Var(h'\delta\hat{\hat{\beta}}) = Var(h'\delta\hat{\beta}) = \sigma^2_h$. If $h \in \mathcal{M}\left[\left(P^C_{C^{-1}G'}\right)'\right]$, then $Var(h'\delta\hat{\hat{\beta}}) = 0$. It follows from (16) and from the fact that $M^C_{C^{-1}G'}M^C_{C^{-1}G'} = M^C_{C^{-1}G'}$ and $P^C_{C^{-1}G'}M^C_{C^{-1}G'} = 0$.

Remark 2.2.10 (Geometrical interpretation of the expression (18)). The definition of the constraints curvature has a natural geometrical meaning. Consider model $Y - f_0 \sim_n (F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \Sigma)$ with the constraints $G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0$, where β_0 is an approximative value of the actual value of the parameter β such that $g(\beta_0) = 0$.

The BLUE of β within the linear model $Y - f_0 \sim_n (F\delta\beta, \Sigma)$ without constraints is

$$\delta\hat{\beta}(Y, \beta_0) = C^{-1}F'\Sigma^{-1}(Y - f_0),\ Var(\delta\hat{\beta}) = C^{-1}.$$

Let us consider the metric space (R^k, C) with the Mahalanobis metric given by $Var^{-1}(\delta\hat{\beta}) = C$. The manifold $\{\delta\beta : G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0\}$ lies in this space. The tangential space to this manifold at the point β_0 is $\mathcal{M}(M^C_{C^{-1}G'})$. An infinitesimal shift δs along the manifold $\{\delta\beta : G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0\}$ causes its deflection $P^C_{C^{-1}G'}\delta s$ from the tangential space (in the chosen direction). Thus the curvature in this direction is

$$K_{\delta s} = \frac{2||P^C_{C^{-1}G'}\delta s||_C}{||M^C_{C^{-1}G'}\delta s||^2_C}.$$

Since $G\delta s + \frac{1}{2}\gamma_{\delta s} = 0$, we can write (by multiplying this condition by $C^{-1}G'(GC^{-1}G')^{-1}$ from the left)

$$P^C_{C^{-1}G'}\delta s = -\frac{1}{2}C^{-1}G'(GC^{-1}G')^{-1}\gamma_{K_G\delta s},$$

thus

$$K_{\delta s}(\beta_0) = \frac{2\sqrt{\frac{1}{4}\gamma'_{K_G\delta s}(GC^{-1}G')^{-1}GC^{-1}CC^{-1}G'(GC^{-1}G')^{-1}\gamma_{K_G\delta s}}}{\delta s'[C - G'(GC^{-1}G')^{-1}G]\delta s}$$

and the curvature $K_I^{(constr)}(\beta_0)$ of the condition is

$$K_I^{(constr)}(\beta_0) = \sup\left\{\frac{\sqrt{\gamma'_{K_G\delta s}(GC^{-1}G')^{-1}\gamma_{K_G\delta s}}}{\delta s' K'_G C K_G \delta s},\ \delta s \in R^{k-q}\right\}.$$

$K_I^{(constr)}(\cdot)$, $\beta \in \{u : g(u) = 0\}$, within the quadratic model $Y - f_0 \sim_n (F\delta\beta + \frac{1}{2}\kappa_{\delta\beta},\ \Sigma)$ with the condition $G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0$, defined by (18) is an analogy of the intrinsic curvature $K^{(int)}(\cdot), \beta \in R^k$, defined by (2) within the model $Y - f_0 \sim_n (F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \Sigma)$. The only difference is that when defining the intrinsic curvature we operate in the linear space R^n with the Mahalanobis metric given by $\Sigma^{-1} = Var^{-1}(Y)$ and we project on the tangential space $\mathcal{M}(F)$ of the mean value surface $E(Y) = f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}$ at the point β_0 and when defining the constraints curvature we operate in the linear space R^k with the Mahalanobis metric given by $C = Var^{-1}(\hat{\beta})$ and we project on the tangential space $\mathcal{M}(M^C_{C^{-1}G'})$ of the manifold $G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0$ at the point β_0 such that $g(\beta_0) = 0$.

Remark 2.2.11 Relations (19), (20), (24) and (26) can serve for determining various linearization regions of a concrete linear function of the estimator $\delta\hat{\beta}(Y, \beta_0)$, i.e. of those regions of changes of the parameter $\delta\beta$ that do not cause a larger deflection in the linear function of the bias than the chosen ε-multiple of the value $\sqrt{h'C^{-1}h}$.

Example 2.2.12 The coordinates y of a straight line $y = \beta_1 + \beta_2 x$ were measured at points $x_1 = -1, x_2 = 0$ and $x_3 = 1$ with the accuracy characterized by the standard deviation σ; the measurements were stochastically independent. The unknown parameters β_1 and β_2 have to fulfil the condition $\beta_1^2 + \beta_2^2 - 2\beta_2 R = 0$ $(R > 0)$. Thus we operate within model

$$Y \sim_3 \left[\begin{pmatrix} 1, & -1 \\ 1, & 0 \\ 1, & 1 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_1 \end{pmatrix}, \sigma^2 I\right],$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \left\{\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} : \beta_1^2 + \beta_2^2 - 2\beta_2 R = 0, R > 0\right\}.$$

The problem is to determine the linearization region for parameter β.

Let the approximate value of β be $\beta_0 = (0,0)'$ (it means that $g(0) = 0$). Since

$$\frac{\partial g(\beta_1,\beta_2)}{\partial \beta_1} = 2\beta_1, \quad \frac{\partial g(\beta_1,\beta_2)}{\partial \beta_2} = 2\beta_2 - 2R,$$
$$\frac{\partial^2 g(\beta_1,\beta_2)}{\partial \beta_1^2} = 2, \quad \frac{\partial^2 g(\beta_1,\beta_2)}{\partial \beta_1 \partial \beta_2} = 0,$$
$$\frac{\partial^2 g(\beta_1,\beta_2)}{\partial \beta_1 \partial \beta_2} = 0, \quad \frac{\partial^2 g(\beta_1,\beta_2)}{\partial b_2^2} = 2,$$

at the point $\beta_0 = (0,0)'$ we get

$$G = (0, -2R),\ \gamma = (\delta\beta_1, \delta\beta_2)\begin{pmatrix} 2, & 0 \\ 0, & 2 \end{pmatrix}\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix} = 2(\delta\beta_1^2 + \delta\beta_2^2),$$

$$C = \frac{1}{\sigma^2}\begin{pmatrix} 3, & 0 \\ 0, & 2 \end{pmatrix},\ C^{-1} = \sigma^2\begin{pmatrix} 1/3, & 0 \\ 0, & 1/2 \end{pmatrix},$$

$$G^-_{m(C)} = C^{-1}G'(GC^{-1}G')^{-1} = \begin{pmatrix} 0 \\ -1/(2R) \end{pmatrix}.$$

The matrix K_G may be chosen in the form $K_G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, thus according Remark 2.2.4

$$\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\delta u + \frac{1}{2}\begin{pmatrix} 0 \\ 1/(2R) \end{pmatrix} 2\delta u^2 = \begin{pmatrix} \delta u \\ R\delta u^2/2 \end{pmatrix} \quad (\delta u = \delta\beta_1).$$

Further, according to (18) (see also Remark 2.2.10)

$$K^{(constr)}(0) = \sup\left\{ \frac{\sqrt{\gamma'_{K_G\delta u}(GC^{-1}G')^{-1}\gamma_{K_G\delta u}}}{\delta u' K_G' C K_G \delta u} : \delta u \in R^1 \right\} = \frac{\sqrt{2}\sigma}{3R}.$$

Since $\kappa_{\delta\beta} = 0$, $C^{(par)}_{I,\delta\beta}(0) = K^{(constr)}(0)$ (compare (25) and (18)). According to Theorem 2.2.8 we get

$$\delta u' K_G' C K_G \delta u \le \frac{2\varepsilon}{C^{(par)}_{I,\delta\beta}} = 3\sqrt{2}\varepsilon\frac{R}{\sigma} = 4.243\frac{\varepsilon R}{\sigma}, \tag{27}$$

that implies

$$\forall\{h \in R^2\}|h'b| \le \varepsilon\sqrt{h'C^{-1}h} = \varepsilon\sigma\sqrt{\frac{h_1^2}{3} + \frac{h_2^2}{2}}.$$

Since in our case $\delta u' K_G' C K_G \delta u = 3\delta u^2 = 3\delta\beta_1^2$, the inequality

$$\delta u' K_G' C K_G \delta u \le 3\sqrt{2}\varepsilon\frac{R}{\sigma},$$

(see the relation (27)) immediately implies that the linearization interval is given by the inequality

$$|\delta\beta_1| \leq \sqrt{\sqrt{2}}\sqrt{\varepsilon\sigma R} = 1.189\sqrt{\varepsilon\sigma R}.$$

The last relation for the given values of ε, σ and R enable us tu judge if the a priori information on the parameter β allows to use the linearized model. The vector $\delta\beta = \beta - \beta_0$ with the accuracy till the second order values is characterized by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}\delta\beta_1$, that means that before the experiment we must know whether $\delta\beta_1$ is significantly smaller than the number $1.189\sqrt{\varepsilon\sigma R}$. The standard deviation of the estimator $\delta\hat{\beta}_1$ in the linearized model equals $\sigma/\sqrt{3}$ and according to the Chebyshev inequality the value $\delta\beta_1$ (with exception of the higher degree values) with a high probability lies in the interval $[\delta\hat{\beta}_1 - 3\sigma/\sqrt{3}, \delta\hat{\beta}_1 + 3\sigma/\sqrt{3}]$. If

$$[\delta\hat{\beta}_1 - 3\sigma/\sqrt{3}, \delta\hat{\beta}_1 + 3\sigma/\sqrt{3}] \subset [-1.189\sqrt{\varepsilon\sigma R}, 1.189\sqrt{\varepsilon\sigma R}],$$

the linearization in entitled.

Let $\delta\hat{\beta}_1 = 0$, then the required inclusion is fulfilled if

$$3\sigma/\sqrt{3} << 1.189\sqrt{\varepsilon\sigma R},$$

i.e. if

$$R >> \frac{3\sigma}{\varepsilon\sqrt{2}} = 2.121\frac{\sigma}{\varepsilon}.$$

Example 2.2.13 The problem is to determine the coordinates ϕ, λ of a point P on a sphere with radius R. The point lies on the curve $(\phi - \phi_0 - \rho)^2 + \lambda^2 = \rho^2$, where ϕ_0 is the approximate value of the coordinate ϕ at the point P and $\rho > 0$ is a known number. The observation vector is three-dimensional and

$$E\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} R\cos\phi\cos\lambda \\ R\cos\phi\sin\lambda \\ R\sin\phi \end{pmatrix}, Var(Y) = \sigma^2 I.$$

The approximate value of the coordinate λ is $\lambda_0 = 0$. The model in our case is described by the relations

$$F = \begin{pmatrix} -R\sin\phi_0, & 0 \\ 0, & R\cos\phi_0 \\ R\cos\phi_0, & 0 \end{pmatrix}, \ \kappa_{\delta\beta} = -R\begin{pmatrix} (\delta\phi^2 + \delta\lambda^2)\cos\phi_0 \\ \delta\phi\delta\lambda 2\sin\phi_0 \\ \delta\phi^2\sin\phi_0 \end{pmatrix},$$

$$g(\phi, \lambda) = (\phi - \phi_0 - \rho)^2 + \lambda^2 - \rho^2$$

$$\frac{\partial g(\phi,\lambda)}{\partial(\phi,\lambda)}\Big|_{\phi_0,0} = (-2\rho, 0),\ \gamma_{(\delta\phi,\delta\lambda)'} = \frac{1}{2}(\delta\phi,\delta\lambda)\begin{pmatrix} 2, & 0 \\ 0, & 2 \end{pmatrix}\begin{pmatrix} \delta\phi \\ \delta\lambda \end{pmatrix} = \delta\phi^2 + \delta\lambda^2.$$

The condition

$$(-2\rho, 0)\begin{pmatrix} \delta\phi \\ \delta\lambda \end{pmatrix} + \frac{1}{2}(\delta\phi,\delta\lambda)\begin{pmatrix} 2, & 0 \\ 0, & 2 \end{pmatrix}\begin{pmatrix} \delta\phi \\ \delta\lambda \end{pmatrix} = 0$$

can be rewriten to the form

$$(1,0)\begin{pmatrix} \delta\phi \\ \delta\lambda \end{pmatrix} + \frac{1}{2}(\delta\phi,\delta\lambda)\begin{pmatrix} -1/\rho, & 0 \\ 0, & -1/\rho \end{pmatrix}\begin{pmatrix} \delta\phi \\ \delta\lambda \end{pmatrix} = 0,$$

thus $G = (1,0)$; if we choose K_G in the form $K_G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $\gamma_{K_G\delta u} = -\frac{1}{\rho}\delta u^2$. Thus directly $\delta\lambda$ can be chosen for δu.

Furthermore

$$C = \frac{1}{\sigma^2}\begin{pmatrix} R^2, & 0 \\ 0, & R^2\cos^2\phi_0 \end{pmatrix},\ C^{-1} = \frac{\sigma^2}{R^2}\begin{pmatrix} 1, & 0 \\ 0, & 1/\cos^2\phi_0 \end{pmatrix},$$

$$P_F^{\Sigma^{-1}} = \begin{pmatrix} \sin^2\phi_0, & 0, & -\sin\phi_0\cos_0 \\ 0, & 1, & 0 \\ -\sin\phi_0\cos\phi_0, & 0, & \cos^2\phi_0 \end{pmatrix},$$

$$M_F^{\Sigma^{-1}} = \begin{pmatrix} \cos^2\phi_0, & 0, & \sin\phi_0\cos\phi_0 \\ 0, & 0, & 0 \\ \sin\phi_0\cos\phi_0, & 0, & \sin^2\phi_0 \end{pmatrix}$$

If the condition is not respected in this model, the Bates and Watts measures of nonlinearity are

$$K^{(int)}(0,\phi_0) =$$
$$= \sup\left\{\frac{\sqrt{\kappa'_{\delta\beta}\Sigma^{-1}M_F^{\Sigma^{-1}}\kappa_{\delta\beta}}}{\delta\beta' C\delta\beta} : \delta\beta = \begin{pmatrix} \delta\phi \\ \delta\lambda \end{pmatrix} \in R^2\right\} =$$
$$= \sup\left\{\frac{\sqrt{\frac{R^2}{\sigma^2}(\delta\phi^4 + 2\cos^2\phi_0\delta\phi^2\delta\lambda^2 + \cos^4\phi_0\delta\lambda^4)}}{\frac{R^2}{\sigma^2}(\delta\phi^2 + \delta\lambda^2\cos^2\phi_0)} : \begin{pmatrix} \delta\phi \\ \delta\lambda \end{pmatrix} \in R^2\right\} = \frac{\sigma}{R}$$

and

$$K^{(par)}(0,\phi_0) = \sup\left\{\frac{\sqrt{\kappa'_{\delta\beta}\Sigma^{-1}P_F^{\Sigma^{-1}}\kappa_{\delta\beta}}}{\delta\beta' C\delta\beta} : \delta\beta = \begin{pmatrix} \delta\phi \\ \delta\lambda \end{pmatrix} \in R^2\right\} =$$
$$= \sup\left\{\frac{\sigma}{R}\sin\phi_0\frac{\sqrt{4\delta\phi^2\delta\lambda^2 + \delta\lambda^4\cos^2\phi_0}}{\delta\phi^2 + \delta\lambda^2\cos^2\phi_0} : \begin{pmatrix} \delta\phi \\ \delta\lambda \end{pmatrix} \in R^2\right\} = \frac{2\sigma}{\sqrt{3}R}\tan\phi_0.$$

If the constraints are taken into account, then

$$K_I^{(constr)}(0,\phi_0) = \sup\left\{\frac{\sqrt{\gamma'_{K_G\delta\lambda}(GC^{-1}G')^{-1}\gamma_{K_G\delta\lambda}}}{\delta\lambda K'_G C K_G \delta\lambda} : \delta\lambda \in R^1\right\} =$$

$$= \sup\left\{\frac{\sqrt{\frac{\delta\lambda^2}{\rho}\frac{R^2}{\sigma^2}\frac{\delta\lambda^2}{\rho}}}{\delta\lambda^2\frac{R^2}{\sigma^2}\cos^2\phi_0} : \delta\lambda \in R^1\right\} = \frac{1}{\rho}\frac{\sigma}{R\cos^2\phi_0}$$

and $K_{I,\delta\beta}^{(par)}(0,\phi_0) = K_I^{(constr)}(0,\phi_0)$, since

$$\kappa'_{K_G\delta u}\Sigma^{-1}F(M_{G'}CM_{G'})^{+}F'\Sigma^{-1}\kappa_{K_G\delta u} = 0.$$

The linearization region such that

$$\forall\{h\in R^2\}|h'b_I| \le \varepsilon\sqrt{h'C^{-1}h} = \varepsilon\frac{\sigma}{R}\sqrt{h_1^2 + \frac{h_2^2}{\cos^2\phi_0}},$$

where $h = (h_1, h_2)'$, $\quad b_I = E(\delta\hat{\hat{\beta}}) - \delta\beta$, has the form

$$\left\{\begin{pmatrix}\delta\phi\\ \delta\lambda\end{pmatrix} : \begin{pmatrix}\delta\phi\\ \delta\lambda\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}\delta\lambda, (\delta\phi,\delta\lambda)C\begin{pmatrix}\delta\phi\\ \delta\lambda\end{pmatrix} \le \frac{2\varepsilon}{K_{I,\delta\beta}^{(par)}}\right\} =$$

$$= \left\{\begin{pmatrix}\delta\beta\\ \delta\lambda\end{pmatrix} : \begin{pmatrix}\delta\beta\\ \delta\lambda\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}\delta\lambda, \quad |\delta\lambda| < \sqrt{\frac{2\varepsilon\rho\sigma}{R}}\right\}.$$

The linearization is possible if the linearization region covers the region in that the vector $(\delta\phi,\delta\lambda)'$ occurs with a practical certainty. If the t-multiple of the standard deviation $\sqrt{Var(\delta\hat{\lambda})} = \sigma/R\cos\phi_0$ (in the model without condition) or $\sqrt{Var(\delta\hat{\hat{\lambda}})}$ (in the model with condition), where t is a sufficiently large number (e.g. if the observation vector is normally distributed for $t = 1.96$ the probability 0.95 is obtained for coverring the true value of $\delta\lambda$ by the interval $[\delta\hat{\lambda} - 1.96\sigma/R\cos\phi_0, \delta\hat{\lambda} + 1.96\sigma/R\cos\phi_0]$), then

$$t\frac{\sigma}{R\cos\phi_0} << \sqrt{\frac{2\varepsilon\rho}{R}}\sqrt{\sigma}$$

and from this we see that for σ

$$\sigma << \frac{2\rho R\cos^2\phi_0}{t^2}\varepsilon$$

must hold.

With respect to the Chebyshev inequality even in the case the observation vector is not normally distributed it suffices that t lies in the interval $[3, 5]$.

2.3 The model with constraints of the type II

Consider the quadratic model $Y - f_0 \sim_n \left(F\delta\beta_1 + \frac{1}{2}\kappa_{\delta\beta_1}, \Sigma\right)$ with the constraints $h(\beta_1, \beta_2) = 0$ which is given in the quadratic form

$$h_0 + H_1\delta\beta_1 + H_2\delta\beta_2 + \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)'} = 0,$$

where

$$h_0 = h(\beta_0),\ H_1 = \frac{\partial h(\beta)}{\partial \beta_1}\Big|_{\beta=\beta_0},\ H_2 = \frac{\partial h(\beta)}{\partial \beta_2}\Big|_{\beta=\beta_0},$$

where $\beta_0 = \begin{pmatrix} \beta_{1,0} \\ \beta_{2,0} \end{pmatrix}$ and

$$\{\omega_{(\delta\beta_1,\delta\beta_2)}\}_i = (\delta\beta_1', \delta\beta_2') \begin{pmatrix} \frac{\partial^2 h_i(\beta)}{\partial\beta_1\partial\beta_1'}\Big|_{\beta=\beta_0}, & \frac{\partial^2 h_i(\beta)}{\delta\beta_1\delta\beta_2'}\Big|_{\beta=\beta_0} \\ \frac{\partial^2 h_i(\beta)}{\partial\beta_2\partial\beta_1'}\Big|_{\beta=\beta_0}, & \frac{\partial^2 h_i(\beta)}{\partial\beta_2\partial\beta_2'}\Big|_{\beta=\beta_0} \end{pmatrix} \begin{pmatrix} \partial\beta_1 \\ \partial\beta_2 \end{pmatrix},$$

$i = 1, \dots, q$.

The vector $\delta\beta_2$ is assumed to be a continuous function of $\delta\beta_1$ with continuous first and second derivatives.

Lemma 2.3.1 Within the model

$$Y - f_0 \sim_n (F\delta\beta_1, \Sigma), \begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix} \in \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : h_0 + H_1 u + H_2 v = 0 \right\},$$

$r(F) = k < n$, $r(H_1, H_2) = q < k_1 + k_2$, $r(H_2) = k_2 < q$, the BLUE of $\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix}$ is

$$\delta\hat{\hat{\beta}}_1 = \delta\hat{\beta}_1 - C^{-1}H_1'[M_{H_2}(H_1C^{-1}H_1')M_{H_2}]^+(h_0 + H_1\delta\hat{\beta}_1), \tag{28}$$

$$\delta\hat{\hat{\beta}}_2 = -[(H_2')^-_{m(H_1C^{-1}H_1')}]'(h_0 + H_1\delta\hat{\beta}_1), \tag{29}$$

where $\delta\hat{\beta}_1 = C^{-1}F'\Sigma^{-1}(Y - f_0)$ is the BLUE within the model $Y - f_0 \sim_n (F\delta\beta_1, \Sigma)$ without constraints; its covariance matrix is

$$Var\begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix} =$$

$$= \begin{pmatrix} (M_{H_1'M_{H_2}}CM_{H_1'M_{H_2}})^+, & -C^{-1}H_1'(H_2')^-_{m(H_1C^{-1}H_1')} \\ -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1C^{-1}, & [H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}H_2]^{-1} - I \end{pmatrix}.$$

Proof. $\delta\hat{\hat{\beta}}_1$ and $\delta\hat{\hat{\beta}}_2$ minimize the function

$$\phi(\delta\beta_1, \delta\beta_2) = (Y - F\delta\beta_1)'\Sigma^{-1}(Y - F\delta\beta_1)$$

under the side condition $h_0 + H_1\delta\beta_1 + H_2\delta\beta_2 = 0$. The auxiliary Lagrange function is

$$\Phi(\delta\beta_1, \delta\beta_2) = (Y - F\delta\beta_1)'\Sigma^{-1}(Y - F\delta\beta_1) - 2\lambda'(h_0 + H_1\delta\beta_1 + H_2\delta\beta_2),$$

where λ is the q-dimensional vector of the Lagrange multipliers. The first of the system of equations

$$\left(\frac{\partial\Phi}{\partial\delta\beta_1} =\right) \qquad 2F'\Sigma^{-1}F\delta\beta_1 - 2F'\Sigma^{-1}Y - 2H_1'\lambda = 0$$
$$\left(\frac{\partial\Phi}{\partial\delta\beta_2} =\right) \qquad -2H_2'\lambda = 0$$

has a solution of the form

$$\delta\beta_1 = C^{-1}F'\Sigma^{-1}Y + C^{-1}H_1'\lambda;$$

the side condition after substituing in it $\delta\beta_1$ of the above form reads

$$h_0 + H_1C^{-1}F'\Sigma^{-1}Y + H_1C^{-1}H_1'\lambda + H_2\delta\beta_2 = 0,$$

thus we obtain the equation

$$\begin{pmatrix} H_1C^{-1}H_1', & H_2 \\ H_2', & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \delta\beta_2 \end{pmatrix} = \begin{pmatrix} -(h_0 + H_1\delta\hat{\beta}_1) \\ 0 \end{pmatrix}.$$

Since $r(H_1, H_2) = q$ and $r(H_2) = k_2$ (the dimension of β_2), there exists the inverse

$$\begin{pmatrix} H_1C^{-1}H_1', & H_2 \\ H_2', & 0 \end{pmatrix}^{-1} =$$
$$= \begin{pmatrix} (M_{H_2}H_1C^{-1}H_1'M_{H_2})^+, & (H_2')^-_{m(H_1C^{-1}H_1')} \\ ((H_2')^-_{m(H_1C^{-1}H_1')})', & -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1C^{-1}H_1'(H_2')^-_{m(H_1C^{-1}H_1')} \end{pmatrix};$$

see [5, proof of Theorem 3.7]. By this the proof can be finished in an obvious way. ♠

Lemma 2.3.2 Consider the model

$$Y - f_0 \sim_n (F\delta\beta_1, \Sigma),\ h_0 + H_1\delta\beta_1 + H_2\delta\beta_2 = 0.$$

The BLUE of β_1 in this model equals the BLUE in the model

$$Y - f_0 \sim_n (F\delta\beta_1, \Sigma),\ M_{H_2}h_0 + M_{H_2}H_1\delta\beta_1 = 0.$$

Proof. According to Lemma 2.3.1

$$\delta\hat{\hat{\beta}}_1 = [I - C^{-1}H_1(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+H_1]\delta\hat{\beta}_1 - C^{-1}(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+h_0;$$

according to Lemma 2.2.1 for $G = M_{H_2}H_1$ and $g_0 = M_{H_2}h_0$

$$\begin{aligned}\delta\hat{\hat{\beta}}_1 &= [I - C^{-1}H_1M_{H_2}(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+M_{H_2}]\delta\hat{\beta}_1 \\ &\quad -C^{-1}H_1'M_{H_2}(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+M_{H_2}h_0;\end{aligned}$$

if the equalities

$$\begin{aligned}M_{H_2}(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+M_{H_2} &= (M_{H_2}H_1C^{-1}H_1'M_{H_2})^+M_{H_2} = \\ = M_{H_2}(H_{H_2}H_1C^{-1}H_1'M_{H_2})^+ &= (M_{H_2}H_1C^{-1}H_1'M_{H_2})^+\end{aligned}$$

are taken into account, the statement is obvious. ♠

Remark 2.3.3 The model $Y - f_0 \sim_n \left(F\delta\beta_1 + \frac{1}{2}\kappa_{\delta\beta_1}, \Sigma\right)$ with the constraints $H_1\delta\beta_1 + H_2\delta\beta_2 + \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} = 0$ is identical with the model

$$Y - f_0 \sim_n \left[F\left(K_{H_1}\delta s - \frac{1}{2}T\omega_{K_H\delta s}\right) + \frac{1}{2}\kappa_{K_{H_1}\delta s}, \Sigma\right],\ \delta s \in R^{k_1+k_2-q}, \quad (30)$$

without constraints. Here $K_H = \begin{pmatrix} K_{H_1} \\ K_{H_2} \end{pmatrix}$ is a kernel matrix of the matrix (H_1, H_2), δs is a new $(k_1 + k_2 - q)$-dimensional parameter arizing from the linearized condition $H_1\delta\beta_1 + H_2\delta\beta_2 = 0$ ($h_0 = h(\beta_0) \equiv 0$, the condition must be fulfilled by the approximate value β_0 of the parameter β) whose nontrivial solution is of the form

$$\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix} = \begin{pmatrix} K_{H_1} \\ K_{H_2} \end{pmatrix}\delta s - \frac{1}{2}T\omega_{K_H\delta s},\ \delta s \in R^{k_1+k_2-q}$$

and

$$T = C^{-1}H_1'(H_1C^{-1}H_1' + H_2H_2')^{-1}.$$

Proof. The unknown quadratic terms $\kappa_{1,\delta s}$ and $\kappa_{2,\delta s}$ such that

$$\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix} = \begin{pmatrix} K_{H_1} \\ K_{H_2} \end{pmatrix}\delta s + \frac{1}{2}\begin{pmatrix} \kappa_{1,\delta s} \\ \kappa_{2,\delta s} \end{pmatrix} \quad (31)$$

are a solution of the equation

$$\frac{1}{2}(H_1, H_2)\begin{pmatrix} \kappa_{1,\delta s} \\ \kappa_{2,\delta s} \end{pmatrix} + \frac{1}{2}\omega_{K_H\delta s} = 0$$

(it arises from the condition $H_1\delta\beta_1 + H_2\delta\beta_2 + \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} = 0$ by substituting here $\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix}$ from (31)). Thus

$$\begin{pmatrix} \kappa_{1,\delta s} \\ \kappa_{2,\delta s} \end{pmatrix} = -(H_1, H_2)^{-}\omega_{K_H\delta s} = -\begin{pmatrix} T \\ U \end{pmatrix}\omega_{K_H\delta s}, \tag{32}$$

where

$$\begin{pmatrix} T \\ U \end{pmatrix} = \begin{pmatrix} C^{-1}H_1'(H_1C^{-1}H_1' + H_2H_2')^{-1} \\ H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1} \end{pmatrix}.$$

♠

Remark 2.3.4 If the model (30) is linearized, then the BLUE of the parameter δs is

$$\delta\hat{s} = (K_{H_1}'F'\Sigma^{-1}FK_{H_1})^{-1}K_{H_1}'F'\Sigma^{-1}(Y - f_0)$$

in a consequence of which

$$\delta\hat{\hat{\beta}}_1 = K_{H_1}\delta\hat{s} = K_{H_1}(K_{H_1}'F'\Sigma^{-1}FK_{H_1})^{-1}K_{H_1}'F'\Sigma^{-1}(Y - f_0).$$

Since $K_{H_1}(K_{H_1}'CK_{H_1})^{-1}K_{H_1}'C = P_{K_{H_1}}^C = M_{C^{-1}H_1'M_{H_2}}^C$

$$\delta\hat{\hat{\beta}}_1 = K_{H_1}\delta\hat{s} = M_{C^{-1}H_1'M_{H_2}}^C C^{-1}F'\Sigma^{-1}(Y - f_0). \tag{33}$$

The estimator (28) for $h_0 = 0$ and the estimator (33) are identical.

Corollary 2.3.5 If

$$Y - f_0 \sim_n \left[F\left(K_{H_1}\delta s - \frac{1}{2}T\omega_{K_H\delta s}\right) + \frac{1}{2}\kappa_{K_{H_1}\delta s},\ \Sigma\right]$$

the bias of the BLUE of the form (33) is

$$b_1 = E(\delta\hat{\hat{\beta}}_1) - \delta\beta_1 = \frac{1}{2}M_{C^{-1}H_1'M_{H_2}}^C C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s} + \frac{1}{2}P_{C^{-1}H_1'M_{H_2}}^C T\omega_{K_H\delta s}. \tag{34}$$

Proof.

$$
\begin{aligned}
E(\delta\hat{\hat{\beta}}_1) &= \\
&= M^C_{C^{-1}H_1'M_{H_2}} C^{-1}F'\Sigma^{-1}\left[F\left(K_{H_1}\delta s - \frac{1}{2}T\omega_{K_H\delta s}\right) + \frac{1}{2}\kappa_{K_{H_1}\delta s}\right] \\
&= M^C_{C^{-1}H_1'M_{H_2}}\left(K_{H_1}\delta s - \frac{1}{2}T\omega_{K_H\delta s}\right) + \frac{1}{2}M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s} \\
&= (I - P^C_{C^{-1}H_1'M_{H_2}})\left(K_{H_1\delta s} - \frac{1}{2}T\omega_{K_H\delta s}\right) + \frac{1}{2}M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s} \\
&= \delta\beta_1 + \frac{1}{2}P^C_{C^{-1}H_1'M_{H_2}}T\omega_{K_H\delta s} + \frac{1}{2}M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s},
\end{aligned}
$$

since $K_{H_1}\delta s - \frac{1}{2}T\omega_{K_H\delta s} = \delta\beta_1$ and $P^C_{C^{-1}H_1'M_{H_2}}K_{H_1} = (I - P^C_{K_{H_1}})K_{H_1} = 0$. ♠

Definition 2.3.6 Within the model $Y - f_0 \sim_n \left(F\delta\beta_1 + \frac{1}{2}\kappa_{\delta\beta_1},\ \Sigma\right)$ with the constraints $H_1\delta\beta_1 + H_2\delta\beta_2 + \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} = 0$, the II-constraints curvature with respect to parameter $\delta\beta_1$ at the point β_0 is defined as

$$
\begin{aligned}
K^{(constr)}_{II,\delta\beta_1} &= \\
&= \sup\left\{\frac{\sqrt{\omega'_{K_H\delta s}T'(P^C_{C^{-1}H_1'M_{H_2}})'CP^C_{C^{-1}H_1'M_{H_2}}T\omega_{K_H\delta s}}}{\delta s'K'_{H_1}CK_{H_1}\delta s} : K_{H_1}\delta s \neq 0\right\}.
\end{aligned}
$$

Theorem 2.3.7 Consider the model $Y - f_0 \sim_n \left(F\delta\beta_1 + \frac{1}{2}\kappa_{\delta\beta_1},\ \Sigma\right)$ with the constraints $H_1\delta\beta_1 + H_2\delta\beta_2 + \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} = 0$. Then

1.
$$
\overline{\delta\beta}_1'C\overline{\delta\beta}_1 \le \frac{2\varepsilon}{K^{(constr)}_{II,\delta\beta_1}(\beta_0)} \tag{35}
$$
where $\overline{\delta\beta}_1 = K_{H_1}\delta s$, implies
$$
\forall\left\{h_1 \in \mathcal{M}[(P^C_{C^{-1}H_1'M_{H_2}})']\right\}\ |h_1'b_1| < \varepsilon\sqrt{h_1'C^{-1}h_1}
$$
and

2.
$$
\overline{\delta\beta}_1'C\overline{\delta\beta}_1 \le \frac{2\varepsilon}{K^{(par)}(\beta_0)}, \tag{36}
$$
where $\overline{\delta\beta}_1 = K_{H_1}\delta s$, implies
$$
\forall\left\{h_1 \in \mathcal{M}[(M^C_{C^{-1}H_1'M_{H_2}})']\right\}\ |h_1'b_1| \le \varepsilon\sqrt{h_1'C^{-1}h_1}.
$$

Proof. 1. Let $h_1 \in \mathcal{M}[(P^C_{C^{-1}H_1'M_{H_2}})']$, then

$$
\begin{aligned}
|h_1'b_1| &= \\
&= \frac{1}{2}\left|u'P^C_{C^{-1}H_1'M_{H_2}}\left(M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s} + P^C_{C^{-1}H_1'M_{H_2}}T\omega_{K_H\delta s}\right)\right| \\
&= \frac{1}{2}\left|h_1'P^C_{C^{-1}H_1'M_{H_2}}T\omega_{K_{H_1}\delta s}\right|;
\end{aligned}
$$

applying the Schwarz inequality we get

$$
\frac{1}{2}\left|h_1'P^C_{C^{-1}H_1'M_{H_2}}T\omega_{K_H\delta s}\right| \leq \sqrt{h_1'C^{-1}h_1}K^{(constr)}_{II,\delta\beta_1}(\beta_0)\frac{1}{2}\delta s'K_{H_1}'CK_{H_1}\delta s,
$$

that using the inequality (35) gives the assertion 1. of the theorem;

2. let $h_1 \in \mathcal{M}[(M^C_{C^{-1}H_1'M_{H_2}})']$; then

$$
\begin{aligned}
|h_1'b_1| &= \\
&= \frac{1}{2}\left|u'M^C_{C^{-1}H_1'M_{H_2}}\left(M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\kappa_{H_{H_1}\delta s} + P^C_{C^{-1}H_1'M_{H_2}}T\omega_{K_H\delta s}\right)\right| \\
&= \frac{1}{2}\left|h_1'C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s}\right|,
\end{aligned}
$$

applying now the Schwarz inequality we get

$$
|h_1'b_1| \leq \sqrt{h_1'C^{-1}h_1}K^{(par)}(\beta_0)\frac{1}{2}\delta s'K_{H_1}'CK_{H_1}\delta s,
$$

that using the inequality (36) gives the assertion 2. of the theorem. ♠

Remark 2.3.8 If $h_1 \in \mathcal{M}[(P^C_{C^{-1}H_1'M_{H_2}})']$, then $Var(h_1'\delta\hat{\hat{\beta}}_1) = 0$.

Definition 2.3.9 The Bates and Watts measures of curvature of the model (30) at the point β_0 are the II-intrinsic curvature

$$
\begin{aligned}
K^{(int)}_{II}(\beta_0) &= \\
&= \sup\left\{\frac{\sqrt{\boxed{1}\,\Sigma^{-1}M^{\Sigma^{-1}}_{FM_{H_1'M_{H_2}}}\boxed{1}}}{\delta s'K_{H_1}'CK_{H_1}\delta s} : K_{H_1}\delta s \neq 0\right\} \\
&= \sup\left\{\frac{\sqrt{\boxed{1}\,\Sigma^{-1}\boxed{1} - \kappa'_{K_{H_1}\delta s}\Sigma^{-1}P^{\Sigma^{-1}}_{FM_{H_1'M_{H_2}}}\kappa_{K_{H_1}\delta s}}}{\delta s'K_{H_1}'CK_{H_1}\delta s} : K_{H_1}\delta s \neq 0\right\},
\end{aligned} \tag{37}
$$

where $\boxed{1} = \kappa_{K_{H_1}\delta s} - FT\omega_{K_H\delta s}$ and the II-parametric curvature

$$
K^{(par)}_{II}(\beta_0) = \sup\left\{\frac{\sqrt{\boxed{1}\,\Sigma^{-1}P^{\Sigma^{-1}}_{FM_{H_1'M_{H_2}}}\boxed{1}}}{\delta s'K_{H_1}'CK_{H_1}\delta s} : K_{H_1}\delta s \neq 0\right\}
$$

$$= \sup\left\{\frac{\sqrt{\kappa'_{K_{H_1}\delta s}\Sigma^{-1}P^{\Sigma^{-1}}_{FM_{H_1'M_{H_2}}}\kappa_{K_{H_1}}\delta s}}{\delta s' K'_{H_1} C K_{H_1}\delta s} : K_{H_1}\delta s \neq 0\right\} \quad (38)$$

(conceive that $P^{\Sigma^{-1}}_{FM_{H_1'M_{H_2}}} FT = 0$ for $T = C^{-1}H_1'(H_1C^{-1}H_1' + H_2H_2')^{-1}$).

Theorem 2.3.10 If

$$K^{(par)}_{II}(\beta_0) = \sup\left\{\frac{\sqrt{\kappa'_{K_{H_1}\delta s}\Sigma^{-1}P^{\Sigma^{-1}}_{FM_{H_1'M_{H_2}}}\kappa_{K_{H_1}\delta s}}}{\delta s' K'_{H_1} C K_{H_1}\delta s} :\ K_{H_1}\delta s \neq 0\right\},$$

then

$$\delta s' K'_{H_1} C K_{H_1}\delta s \le \frac{2\varepsilon}{K^{(par)}_{II}(\beta_0)} \quad (39)$$

implies

$$\forall\left\{h_1 \in \mathcal{M}\left[\left(M^C_{C^{-1}H_1'M_{H_2}}\right)'\right]\right\}\ |h_1'b_1| \le \varepsilon\sqrt{h_1'C^{-1}h_1}.$$

Proof. If $h_1 \in \mathcal{M}\left[\left(M^C_{C^{-1}H_1'M_{H_2}}\right)'\right]$, then

$$\begin{aligned}
&|h_1'b_1| = \\
&= |h_1'M^C_{C^{-1}H_1'M_{H_2}}b_1| \le \sqrt{h_1'C^{-1}h_1}\sqrt{b_1'(M^C_{C^{-1}H_1'M_{H_2}})'CM^C_{C^{-1}H_1'M_{H_2}}b_1} \\
&= \sqrt{h_1'C^{-1}h_1}\frac{1}{2}\sqrt{\kappa'_{K_{H_1}\delta s}\Sigma^{-1}FM^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s}} \\
&= \sqrt{h_1'C^{-1}h_1}\frac{1}{2}\sqrt{\kappa'_{K_{H_1}\delta s}\Sigma^{-1}F(M_{H_1'M_{H_2}}CM_{H_1'M_{H_2}})^{+}F'\Sigma^{-1}\kappa_{K_{H_1}}} \\
&= \sqrt{h_1'C^{-1}h_1}\frac{1}{2}\sqrt{\kappa'_{K_{H_1}\delta s}\Sigma^{-1}FM_{H_1'M_{H_2}}(M_{H_1'M_{H_2}}CM_{H_1'M_{H_2}})^{+}M_{H_1'M_{H_2}}F'\Sigma^{-1}\kappa_{K_{H_1}}} \\
&= \sqrt{h_1'C^{-1}h_1}\frac{1}{2}\sqrt{\kappa'_{K_{H_1}\delta s}\Sigma^{-1}P^{\Sigma^{-1}}_{FM_{H_1'M_{H_2}}}\kappa_{K_{H_1}\delta s}} \\
&\le \sqrt{h_1'C^{-1}h_1}K^{(par)}_{II}\frac{1}{2}\delta s' K'_{H_1} C K_{H_1}\delta s.
\end{aligned}$$

For the auxiliary relations see Theorem 2.2.6. This theorem is an alternative to Theorem 2.3.7 2.

Definition 2.3.11 The II-parametric measure of nonlinearity of the model $Y - f_0 \sim_n \left(F\delta\beta_1 + \frac{1}{2}\kappa_{\delta\beta_1},\ \Sigma\right)$ with the constraints $H_1\delta\beta_1 + H_2\delta\beta_2 + \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} = 0$ with respect to the parameter $\delta\beta_1$ at the point β_0 is is defined as

$$C^{(par)}_{II,\delta\beta_1}(\beta_0) = \quad (40)$$

$$= \sup\left\{ \frac{\sqrt{\kappa'_{K_{H_1}\delta s}\Sigma^{-1}P^{\Sigma^{-1}}_{FM_{H'_1M_{H_2}}}\kappa_{K_{H_1}\delta s} + \omega'_{K_H\delta s}T'CP^{C}_{C^{-1}H'_1M_{H_2}}T\omega_{K_H\delta s}}}{\delta s' K'_{H_1}CK_{H_1}\delta s} : \right.$$

$$\left. : K_{H_1}\delta s \neq 0 \right\}.$$

Theorem 2.3.12 If

$$\delta u' K'_{H_1} C K_{H_1} \delta u \leq \frac{2\varepsilon}{C^{(par)}_{II,\delta\beta_1}(\beta_0)}, \tag{41}$$

then

$$\forall \{h_1 \in R^{k_1}\}\ |h'_1 b_1| \leq \varepsilon\sqrt{h'_1 C^{-1} h_1}.$$

Proof. According to (34)

$$|h'_1 b_1| = \frac{1}{2}\left| h'_1 \left(M^{C}_{C^{-1}H'_1M_{H_2}} C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s} + P^{C}_{C^{-1}H'_1M_{H_2}} T\omega_{K_H\delta s}\right)\right|.$$

Applying the Schwarz inequality we get

$$\begin{aligned} |h'_1 b_1| &\leq \\ &\leq \sqrt{h'_1 C^{-1} h_1}\frac{1}{2}\sqrt{\boxed{2}' C \boxed{2}} \\ &= \sqrt{h'_1 C^{-1} h_1}\frac{1}{2}\sqrt{\kappa'_{K_{H_1}\delta s}\Sigma^{-1}P^{\Sigma^{-1}}_{FM_{H'_1M_{H_2}}}\kappa_{K_{H_1}\delta s} + \omega'_{K_H\delta s}T'P^{C}_{C^{-1}H'_1M_{H_2}}T\omega_{K_H\delta s}}, \end{aligned}$$

where

$$\boxed{2} = M^{C}_{C^{-1}H'_1M_{H_2}} C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s} + P^{C}_{C^{-1}H'_1M_{H_2}} T\omega_{K_H\delta s}.$$

Taking now into account relation (40) and the assumption (41) we get the assertion of the theorem.

Here the facts that

$$\begin{aligned} (M^{C}_{C^{-1}H'_1M_{H_2}})' C M^{C}_{C^{-1}H'_1M_{H_2}} &= C M^{C}_{C^{-1}H'_1M_{H_2}}, \\ F M^{C}_{C^{-1}H'_1M_{H_2}} C^{-1}F'\Sigma^{-1} &= P^{\Sigma^{-1}}_{FM_{H'_1M_{H_2}}}, \\ (P^{C}_{C^{-1}H'_1M_{H_2}})' C P^{C}_{C^{-1}H'_1M_{H_2}} &= C P^{C}_{C^{-1}H'_1M_{H_2}} \quad \text{and} \\ (P^{C}_{C^{-1}H'_1M_{H_2}})' C M^{C}_{C^{-1}H'_1M_{H_2}} &= 0 \end{aligned}$$

were taken into account. ♠

In the following we shall deal with the parameter β_2, for which according to (32) it holds $\delta\beta_2 = K_{H_2}\delta s - \frac{1}{2}U\omega_{K_H\delta s}$.

Remark 2.3.13

$$[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1\delta\hat{\hat{\beta}}_1 = [(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1\delta\hat{\beta}_1. \tag{42}$$

Proof. Using (28) for $h_0 = 0$ we can write

$$\delta\hat{\hat{\beta}}_1 = M^C_{C^{-1}H_1'M_{H_2}}\delta\hat{\beta}_1 = [I - C^{-1}H_1'M_{H_2}(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+M_{H_2}H_1]\delta\hat{\beta}_1,$$

thus it suffices to prove that

$$[(H_2')^-_{m(H_1'C^{-1}H_1')}]'H_1C^{-1}H_1'M_{H_2}(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+M_{H_2}H_1 = 0.$$

This equality is true since for

$$[(H_2')^-_{m(H_1C^{-1}H_1')}]' = [H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}H_2]^{-1}H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}$$

it holds

$$\begin{aligned}
&[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1C^{-1}H_1'M_{H_2}(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+M_{H_2}H_1 = \\
&= [H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}H_2]^{-1}H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1} \times \\
&\quad \times(H_1C^{-1}H_1' + H_2H_2')(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+H_1 = 0
\end{aligned}$$

as $H_2'(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+ = 0$. ♠

Theorem 2.3.14

$$\begin{aligned}
b_2 &= E(\delta\hat{\hat{\beta}}_2) - \delta\beta_2 \\
&= \frac{1}{2}[H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}H_2]^{-1}H_2' \times \\
&\quad \times(H_1C^{-1}H_1' + H_2H_2')^{-1}(\omega_{(\delta\beta_1,\delta\beta_2)} - H_1C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta_1}).
\end{aligned} \tag{43}$$

Proof. According to (29) and (42) for $h_0 = 0$

$$\begin{aligned}
\delta\hat{\hat{\beta}}_2 &= -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1\delta\hat{\hat{\beta}}_1 \\
&= -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1M^C_{C^{-1}H_1'M_{H_2}}\delta\hat{\beta}_1 \\
&= -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}(Y - f_0).
\end{aligned}$$

Thus

$$\begin{aligned}
E(\delta\hat{\hat{\beta}}_2) &= -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\left(F\delta\beta_1 + \frac{1}{2}\kappa_{\delta\beta_1}\right) \\
&= -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1C^{-1}F'\Sigma^{-1}\left(F\delta\beta_1 + \frac{1}{2}\kappa_{\delta\beta_1}\right) +
\end{aligned}$$

$$
\begin{aligned}
& +[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1P^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\left(F\delta\beta_1+\frac{1}{2}\kappa_{\delta\beta_1}\right)\\
= \; & -[(H_2')^-_{m(H_1C^{-1}H_1')}]'\left(H_1\delta\beta_1+\frac{1}{2}H_1C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta_1}\right)+\\
& +[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1P^C_{C^{-1}H_1'M_{H_2}}\times\\
& \times\left(K_{H_1}\delta s-\frac{1}{2}T\omega_{K_H\delta s}+\frac{1}{2}C^{-1}F'\Sigma^{-1}\kappa_{K_H\delta s}\right)\\
= \; & -[(H_2')^-_{m(H_1C^{-1}H_1')}]'\left(-H_2\delta\beta_2-\frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)}+\frac{1}{2}H_1C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta_1}\right)+\\
& +\frac{1}{2}[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1P^C_{C^{-1}H_1'M_{H_2}}(-T\omega_{K_H\delta s}+C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s})\\
= \; & \delta\beta_2+\frac{1}{2}[(H_2')^-_{m(H_1C^{-1}H_1')}]'\times\\
& \times(\omega_{K_h\delta s}-H_1P^C_{C^{-1}H_1'M_{H_2}}T\omega_{K_H\delta s}-H_1M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s}).
\end{aligned}
$$

Here the fact that $P^C_{C^{-1}H_1'M_{H_2}}K_{H_1}=0$ was taken into account ($P^C_{C^{-1}H_1'M_{H_2}}=I-M^C_{C^{-1}H_1'M_{H_2}}=I-P^C_{K_{H_1}}$ and $P^C_{K_{H_1}}K_{H_1}=K_{H_1}$). Now it is necessary to take into account that

$$
[(H_2')^-_{m(H_1C^{-1}H_1')}]'=[H_2'(H_1C^{-1}H_1'+H_2H_2')^{-1}H_2]^{-1}H_2'(H_1C^{-1}H_1'+H_2H_2')^{-1}
$$

and that $H_2'M_{H_2}=0$. Using this we obtain

$$
\begin{aligned}
& [(H_2')^-_{m(H_1C^{-1}H_1')}]'(\omega_{K_H\delta s}-H_1P^C_{C^{-1}H_1'M_{H_2}}T\omega_{K_H\delta s}-H_1M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s})=\\
& = \; [H_2'(H_1C^{-1}H_1'+H_2H_2')^{-1}H_2]^{-1}H_2'(H_1C^{-1}H_1'+H_2H_2')^{-1}\times\\
& \quad \times\big(\omega_{K_H\delta s}-H_1C^{-1}H_1'M_{H_2}(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+M_{H_2}H_1T\omega_{K_H\delta s}+\\
& \quad +H_1[I-C^{-1}H_1'M_{H_2}(M_{H_2}H^{-1}C^{-1}H_1'M_{H_2})^+M_{H_2}H_1]C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s}\big)\\
& = \; [H_2'(H_1C^{-1}H_1'+H_2H_2')^{-1}H_2]^{-1}H_2'(H_1C^{-1}H_1'+H_2H_2')^{-1}\times\\
& \quad \times(\omega_{(\delta\beta_1,\delta\beta_2)}-H_1C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta_1}),
\end{aligned}
$$

by which the theorem is proved. ♠

The rank of the $k_1\times(k_1+k_2-q)$ matrix K_{H_1} is $r(K_{H_1})=k_1+k_2-q$. It follows from the assumptions $r(H_1,H_2)=q<k_1+k_2$ & $r(H_2)=k_2<q$ and from the relation

$$
K_{H_2}=-[(H')^-_{m(H_1'C^{-1}H_1)}]'H_1K_{H_1},
$$

which implies that the rows of the matrix K_{H_2} are a linear combination of the rows of the matrix K_{H_1}, therefore

$$
k_1+k_2-q=r\begin{pmatrix}K_{H_1}\\ K_{H_2}\end{pmatrix}=r(K_{H_1}).
$$

Thus between the vectors δu and $K_{H_1}\delta u$ there exists a one-to-one relation.

Generally, since K_{H_2} is of the type $k_2 \times (k_1 + k_2 - q)$ a one-to-one relation between the vectors δu and $K_{H_2}\delta u$ cannot exist. This fact causes difficulties in defining the curvature measures for the parameter $\delta\beta_2$ within the model

$$-H_1\delta\hat{\beta}_1 \sim_q \left(H_2\delta\beta_2 + \frac{1}{2}(\omega_{K_H\delta s} - H_1C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta s}), H_1C^{-1}H_1'\right). \quad (44)$$

Moreover the covariance matrix $H_1C^{-1}H_1'$ in this model is singular.

Thus the basis for constructing the linearization region is the following definition of nonlinearity measure for $\delta\beta_2$.

Definition 2.3.15 Let

$$C^{(par)}_{II,\delta\beta_2}(\beta_0) = \quad (45)$$

$$= \sup\left\{\frac{\sqrt{\boxed{3}'(H_1C^{-1}H_1' + H_2H_2')^{-1}P_{H_2}^{(H_1C^{-1}H_1'+H_2H_2')^{-1}}\boxed{3}}}{\delta u' K_{H_1}' C K_{H_1}\delta u} : \delta u \in R^{k_1+k_2-q}\right\},$$

where $\boxed{3} = \omega_{K_H\delta u} - H_1C^{-1}F'\Sigma^{-1}\kappa_{K_{H_1}\delta u}$.

Theorem 2.3.16 If

$$\delta u' K_{H_1}' C K_{H_1}\delta u < \frac{2\varepsilon}{C^{(par)}_{II,\delta\beta_2}(\beta_0)},$$

then

$$\forall\{h_2 \in R^{k_2}\}\ |h_2'b_2| \leq \varepsilon\sqrt{h_2'[H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}H_2]^{-1}h_2}.$$

Proof. According to (43)

$$\begin{aligned}|h_2'b_2| &= \frac{1}{2}|h_2'[H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}H_2]^{-1} \times \\ &\quad \times H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}(\omega_{(\delta\beta_1,\delta\beta_2)} - H_1C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta_1})|.\end{aligned}$$

By using the Schwarz inequality we get

$$\begin{aligned}|h_2'b_2| &\leq \frac{1}{2}\sqrt{h_2'[H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}H_2]^{-1}h_2} \times \\ &\quad \times\Big((\omega_{(\delta\beta_1,\delta\beta_2)} - H_1C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta_1})'(H_1C^{-1}H_1' + H_2H_2')^{-1}H_2 \times \\ &\quad \times[H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}H_2]^{-1} \times \\ &\quad \times H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}(\omega_{(\delta\beta_1,\delta\beta_2)} - H_1C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta_1})\Big)^{1/2} \\ &= \frac{1}{2}\sqrt{h_2'[H_2'(H_1C^{-1}H_1' + H_2H_2')^{-1}H_2]^{-1}h_2} \times \\ &\quad \times\Big((\omega_{(\delta\beta_1,\delta\beta_2)} - H_1C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta_1})(H_1C^{-1}H_1' + H_2H_2')^{-1} \times \\ &\quad \times P_{H_2}^{(H_1C^{-1}H_1'+H_2H_2')^{-1}}(\omega_{(\delta\beta_1,\delta\beta_2)} - H_1C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta_1})\Big)^{1/2},\end{aligned}$$

by which the proof is finished. ♠

Remark 2.3.17 The requirements that $g(\beta_0) = 0$ and $h(\beta_{10}, \beta_{20}) = 0$, which are emphasized in subsections concerning the models with constraints of the type I and II, respectively, have to be satisfied from several reasons.

If $g(\beta_0) \neq 0$, the manifold $\{\beta_0 + \delta\beta : g(\beta_0) + G\delta\beta = 0\}$ does not form a tangential space of the manifold $\{\beta : g(\beta) = 0\}$ therefore it does not represent even the approximation of the first order. Moreover the estimator $\delta\hat{\hat{\beta}}$ is a projection of the estimator $\delta\hat{\beta} = C^{-1}F'\Sigma^{-1}(Y - f_0)$ on the linear manifold $\{\delta\beta : g(\beta_0) + G\delta\beta = 0\}$. Therefore the estimator $\hat{\hat{\beta}} = \beta_0 + \delta\hat{\hat{\beta}}$ that would belong to the linear manifold $\{\beta_0 + \delta\beta : g(\beta_0) + G\delta\beta = 0\}$ could not in general satisfy the condition $g(\beta) = 0$ and simultaneously it could be more biased than $\delta\hat{\beta}$. In the case of the linear condition the estimating algorithm ensures that the condition is precisely satisfied and thus the mentioned problem does not occur.

Analogous reason requires the condition $h(\beta_{10}, \beta_{20}) = 0$ to be fulfilled in the case of the constraints of the form $h(\beta_1, \beta_2) = 0$.

2.4 The weak nonlinearity

Till now the notion of the weak nonlinearity of the model has been understood at least intuitively. In the following some quantitative facts are given that enable us to do the notion of a weak nonlinearity more precise.

In Corollary 2.1.2 and Lemma 2.1.3 it was shown that if

$$\delta\beta' C \delta\beta \leq \frac{2\varepsilon}{K^{(par)}(\beta_0)},$$

then

$$\forall \{h \in R^k\} |E(h'\delta\hat{\beta}) - h'\delta\beta| \leq \varepsilon\sqrt{h'C^{-1}h}.$$

Simultaneously in the case of a normally distributed observation vector, the vector $\delta\beta$ with a high probability equalling $1 - \alpha$ is covered by the ellipsoid

$$\{\delta\beta : (\delta\beta - \delta\hat{\beta})' C (\delta\beta - \delta\hat{\beta}) \leq \chi^2_k(0; 1 - \alpha)\}.$$

Thus the model can be considered as a model with a weak nonlinearity at the point β_0 if

$$\chi^2_k(0; 1 - \alpha) << \frac{2\varepsilon}{K^{(par)}(\beta_0)}.$$

For $\varepsilon = 1$, $\alpha = 0.05$ and various k see the following table

k	2	5	10	20	30
$2/\chi_k^2(0;0.95)$	0.334	0.180	0.109	0.064	0.046

It has been proved in Lemma 1.3 that if

$$\delta\beta' C\delta\beta \leq \frac{2\sqrt{\delta_{\max}}}{K^{(int)}(\beta_0)},$$

the risk, of the test of fit the measured data with the model does not increase more than by ε, where

$$P\{\chi^2_{n-k}(\delta_{\max}) \leq \chi^2_{n-k}(0; 1-\alpha)\} = 1 - (\alpha + \varepsilon).$$

For $1 - \alpha = 0.975$ and $\varepsilon = 0.025$ see the following table

$n-k$	2	5	10	20	30
$2\sqrt{\delta_{\max}}/\chi^2_{n-k}(0;0.975)$	0.191	0.140	0.108	0.079	0.063

Thus the model can be considered to be a model with a weak nonlinearity with respect to the fit of data with the model if for corresponding degrees of freedom $K^{(int)}(\beta_0)$ does not exceed the number from the table.

3 The bias of the estimator of the unit dispersion

3.1 The model without constraints

Let us consider the model

$$Y \sim N_n\left(f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta},\ \sigma^2 V\right),\ \delta\beta \in R^k,$$

$r(F) = k < n$, $\sigma^2 \in (0, \infty)$, V a positive definite matrix.

The parameter σ^2 is estimated by the statistic

$$\hat{\sigma}^2 = \frac{1}{n-k}(Y - f_0)'(M_F V M_F)^+(Y - f_0).$$

If we denote $C_0 = F'V^{-1}F$ and

$$K_0^{(int)}(\beta_0) = \sup\left\{\frac{\sqrt{\kappa_{\delta\beta} V^{-1} M_F^{V^{-1}} \kappa_{\delta\beta}}}{\delta\beta' C_0 \delta\beta} : \delta\beta \in R^k\right\}, \tag{46}$$

then obviously

$$K_0^{(int)}(\beta_0) = K^{(int)}(\beta_0)/\sigma.$$

Theorem 3.1.1

1. If

$$\delta\beta' C_0 \delta\beta < \frac{\varepsilon 2\sqrt{n-k}}{K_0^{(int)}(\beta_0)}\sigma,$$

then

$$|E(\hat{\sigma}^2) - \sigma^2| < \varepsilon^2\sigma^2.$$

2. If

$$\delta\beta' C_0 \delta\beta < \frac{\varepsilon 2\sqrt{n-k}}{K_0^{(int)}(\beta_0)}\sigma,$$

then

$$|E(\hat{\sigma}) - \sigma| < \frac{\varepsilon^2}{2}\sigma.$$

Proof. 1. The statistic $\hat{\sigma}^2$ possesses the noncentral $\sigma^2\chi^2_{n-k}(\delta)/(n-k)$ probability distribution, where

$$\delta = \frac{1}{4\sigma^2}\kappa'_{\delta\beta}(M_F V M_F)^+\kappa_{\delta\beta}.$$

Since $E[\chi^2_{n-k}(\delta)] = n - k + \delta$,

$$\begin{aligned} E[\sigma^2\chi^2_{n-k}(\delta)/(n-k)] &= E(\hat{\sigma}^2) = \frac{\sigma^2}{n-k}\left(n - k + \frac{1}{4\sigma^2}\kappa'_{\delta\beta}(M_F V M_F)^+\kappa_{\delta\beta}\right) \\ &= \sigma^2 + \frac{1}{4(n-k)}\kappa'_{\delta\beta}(M_F V M_F)^+\kappa_{\delta\beta}. \end{aligned}$$

The definition of $K_0^{(int)}(\beta_0)$ implies

$$\frac{1}{4(n-k)}\kappa'_{\delta\beta}(M_F V M_F)^+\kappa_{\delta\beta} \le \frac{1}{4(n-k)}\left(K_0^{(int)}(\beta_0)\right)^2(\delta\beta' C_0 \delta\beta)^2.$$

Thus if

$$\delta\beta' C_0 \delta\beta \le \frac{\varepsilon 2\sqrt{n-k}}{K_0^{(int)}(\beta_0)}\sigma,$$

then

$$|E(\hat{\sigma}^2) - \sigma^2| \le \varepsilon^2\sigma^2.$$

2. For $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$ we have

$$\begin{aligned} \sqrt{\hat{\sigma}^2} &= f(\hat{\sigma}^2) \\ &= f(\sigma^2) + \frac{1}{1!}\frac{df(\sigma^2)}{d\sigma^2}(\hat{\sigma}^2 - \sigma^2) + \frac{1}{2!}\frac{d^2 f(\sigma^2)}{d(\sigma^2)^2}(\hat{\sigma}^2 - \sigma^2)^2 + \dots \\ &= \sigma + \frac{1}{2\sqrt{\sigma^2}}(\hat{\sigma}^2 - \sigma^2) + \frac{1}{2!}\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{(\sigma^2)^{3/2}}(\hat{\sigma}^2 - \sigma^2)^2 + \dots, \end{aligned}$$

that implies

$$|E(\hat{\sigma}-\sigma)| \leq \frac{1}{2\sigma}|E(\hat{\sigma}^2)-\sigma^2| + \dots.$$

Now the implication (holding until the terms of the higher degrees)

$$\delta\beta' C_0 \delta\beta \leq \frac{2\sqrt{n-k}\varepsilon}{K_0^{(int)}(\beta_0)}\sigma \Rightarrow |E\hat{\sigma}-\sigma| \leq \frac{\varepsilon^2\sigma}{2}$$

is obvious. ♠

Remark 3.1.2 For handling potential series of a random variable see [6].

3.2 The model with constraints of the type I

Let us consider the model

$$Y \sim N_n\left(f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \sigma^2 V\right),\ G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0,$$

$r(F) = k < n$, V is positive definite, $r(G_{q,k}) = q < k$, $\sigma^2 \in (0,\infty)$. Within this model the parameter σ^2 may be estimated using the statistic

$$\hat{\sigma}^2 = (Y-f_0)'(M_{FM_{G'}} V M_{FM_{G'}})^+ (Y-f_0)/(n+q-k),$$

that respects the condition concerning the parameter β in its linearized form or the statistic

$$\tilde{\sigma}^2 = (Y-f_0)'(M_F V M_F)^+(Y-f_0)/(n-k)$$

that does not respect the condition. The estimator $\hat{\sigma}^2$ has the $\sigma^2\chi^2_{n+q-k}(\delta)/(n+q-k)$ probability distribution whose noncentrality parameter is

$$\delta = E[(Y-f_0)'](M_{FM_{G'}} V M_{FM_{G'}})^+ E(Y-f_0)/\sigma^2, \tag{47}$$

the statistic $\tilde{\sigma}^2$ has the noncentral $\sigma^2\chi^2_{n-k}(\delta_1)/(n-k)$ probability distribution with the noncentrality parameter

$$\delta_1 = E[(Y-f_0)'](M_F V M_F)^+ E(Y-f_0)/\sigma^2.$$

Since $E[\chi^2_f(\delta)] = f+\delta$ and $Var[\chi^2_f(\delta)] = 2f+4\delta$, for $MSE(\hat{\sigma}^2)$ and $MSE(\tilde{\sigma}^2)$ we get the relations

$$\begin{aligned} MSE(\hat{\sigma}^2) &= Var(\hat{\sigma}^2) + [\text{bias}(\hat{\sigma}^2)]^2 \\ &= \sigma^4\left(\frac{2}{n+q-k} + \frac{4E[(Y-f_0)'](M_{FM_{G'}} V M_{FM_{G'}})^+ E(Y-f_0)}{\sigma^2(n+q-k)^2}\right) \\ &\quad + \left(E[(Y-f_0)'](M_{FM_{G'}} V M_{FM_{G'}})^+ E(Y-f_0)/(n+q-k)\right)^2, \end{aligned}$$

and

$$
\begin{aligned}
MSE(\tilde{\sigma}^2) &= Var(\tilde{\sigma}^2) + [\text{bias}(\tilde{\sigma}^2)]^2 \\
&= \sigma^4 \left(\frac{2}{n-k} + \frac{4E[(Y-f_0)'](M_F V M_F)^+ E(Y-f_0)}{\sigma^2 (n-k)^2} \right) \\
&\quad + \Big(E[(Y-f_0)'](M_F V M_F)^+ E(Y-f_0)/(n-k) \Big)^2,
\end{aligned}
$$

where the fact that

$$
\left(M_{FM_{G'}} V M_{FM_{G'}} \right)^+ = (M_F V M_F)^+ + V^{-1} P^{V^{-1}}_{FC_0^{-1}G'}
$$

was taken into account. Thus the situation that $MSE(\tilde{\sigma}^2) < MSE(\hat{\sigma}^2)$ can occur, what cannot occur in the linear model.

Lemma 3.2.1 The bias $E(\hat{\sigma}^2) - \sigma^2$ of the estimator $\hat{\sigma}^2$ is

$$
\begin{aligned}
&E(\hat{\sigma}^2) - \sigma^2 = \\
&= \frac{(\kappa_{K_G\delta u} - FG^-_{m(C_0)}\gamma_{K_G\delta u})'(M_{FM_{G'}} V M_{FM_{G'}})^+ (\kappa_{K_G\delta u} - FG^-_{m(C_0)}\gamma_{K_G\delta U})}{4(n+q-k)}.
\end{aligned}
$$

Proof. Since $\hat{\sigma}^2 \sim \sigma^2 \chi^2_{n+q-k}(\delta)/(n+q-k)$,

$$
E(\hat{\sigma}^2) = \sigma^2 (n+q-k+\delta)/(n+q-k) = \sigma^2 + \frac{\sigma^2 \delta}{n+q-k}.
$$

The noncentrality parameter is given by (47), thus

$$
E(\hat{\sigma}^2) - \sigma^2 = \frac{E[(Y-f_0)'](M_{FM_{G'}} V M_{FM_{G'}})^+ E(Y-f_0)}{n+q-k}.
$$

Furthermore

$$
E(Y - f_0) = F\delta\beta + \frac{1}{2}\kappa_{\delta\beta} \quad \& \quad G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0.
$$

The last equality enables us to express $\delta\beta$ using a new parameter δu (with the accuracy till the second order terms) in the form

$$
\delta\beta = K_G \delta u - G^-_{m(C_0)} \frac{1}{2} \gamma_{K_G \delta u}.
$$

Since $\mathcal{M}(FK_G) = \mathcal{M}(FM_{G'})$ implies that $(M_{FM_{G'}} V M_{FM_{G'}})^+ F K_G \delta u = 0$, we get

$$
\begin{aligned}
&E[(Y-f_0)'](M_{FM_{G'}} V M_{FM_{G'}})^+ E(Y-f_0) = \\
&= \frac{1}{4} (\kappa_{K_G\delta u} - FG^-_{m(C_0)}\gamma_{K_G\delta u})'(M_{FM_{G'}} V M_{FM_{G'}})^+ (\kappa_{K_G\delta u} - FG^-_{m(C_0)}\gamma_{K_G\delta u})
\end{aligned}
$$

(it is to be noticed that each g-inverse G^- possesses full rank in columns).♠

In the following we use the notation

$$K_{I,0}^{(int)}(\beta_0) = \frac{1}{\sigma} K_I^{(int)}(\beta_0);$$

the value

$$K_{I,0}^{(int)}(\beta_0) = \sup\left\{ \frac{\sqrt{\boxed{4}'(M_{FM_{G'}} V M_{FM_{G'}})^+ \boxed{4}}}{\delta s' K_G' F' V^{-1} F K_G \delta s} : \delta s \in R^{k-q} \right\},$$

where $\boxed{4} = \kappa_{K_G\delta s} - FG^-_{m(C_0)}\gamma_{K_G\delta s}$, is independent of σ^2.

Theorem 3.2.2 Let $\delta\beta \in \mathcal{M}(M_{G'}) = \mathcal{M}(K_G)$. If

$$\delta s' K_G' C_0 K_G \delta s \leq \frac{2\sqrt{n+q-k}}{K_{I,0}^{(int)}(\beta_0)} \varepsilon\sigma,$$

then

$$|E(\hat{\sigma}^2) - \sigma^2| \leq \varepsilon^2\sigma^2.$$

Proof. With respect to Lemma 3.2.1

$$E(\hat{\sigma}^2) - \sigma^2 =$$

$$= \frac{(\kappa_{K_G\delta u} - FG^-_{m(C_0)}\gamma_{K_G\delta u})'(M_{FM_{G'}} V M_{FM_{G'}})^+(\kappa_{K_G\delta u} - FG^-_{m(C_0)}\gamma_{K_G\delta u})}{4(n+q-k)},$$

thus

$$E(\hat{\sigma}^2) - \sigma^2 \leq \left(K_{I,0}^{(int)}(\beta_0)\right)^2 \left(\frac{1}{2}\delta s' K_G' F' V^{-1} F K_G \delta s\right)^2 \frac{1}{n+q-k}.$$

Therefore if

$$\delta s' K_G' F' V^{-1} F K_G \delta s \leq \frac{2\sqrt{n+q-k}}{K_{I,0}^{(int)}(\beta_0)} \varepsilon\sigma,$$

the assertion is obvious. ♠

3.3 The model with constraints of the type II

Let us consider the model

$$Y \sim N_n\left(f_0 + F\delta\beta_1 + \frac{1}{2}\kappa'_{\delta\beta_1}, \sigma^2 V\right), \delta\beta = \begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix},$$

$$H_1\delta\beta_1 + H_2\delta\beta_2 + \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} = 0,$$

$r(F) = k_1 < n, \delta\beta_1$ is a k_1-dimensional vector, $\delta\beta_2$ is a k_2-dimensional vector, $r(H_1, H_2) = q < k_1 + k_2$, $r(H_2) = k_2 < q$, V is positive definite and (H_1, H_2) is of the type $q \times (k_1 + k_2)$.

The estimator of the parameter σ^2 is of the form

$$\hat{\sigma}^2 = \frac{(Y - f_0)' \left(M_{FM_{H_1'M_{H_2}}} V M_{FM_{H_1'M_{H_2}}} \right)^+ (Y - f_0)}{n + q - k_1 - k_2};$$

it has a $\sigma^2 \chi^2_{n+q-k_1-k_2}(\delta)/(n + q - k_1 - k_2)$-probability distribution and its parameter of noncentrality is

$$\delta = \frac{1}{\sigma^2} E[(Y - f_0)'] \left(M_{FM_{H_1'M_{H_2}}} V M_{FM_{H_1'M_{H_2}}} \right)^+ E(Y - f_0).$$

Lemma 3.3.1 The bias in the model $Y \sim_n (f_0 + F\delta\beta_1 + \frac{1}{2}\kappa_{\delta\beta_1}, \sigma^2 V)$, $\delta\beta = \begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix}$, $H_1\delta\beta_1 + H_2\delta\beta_2 + \frac{1}{2}\omega_{(\delta\beta_1, \delta\beta_2)} = 0$, is

$$E(\hat{\sigma}^2) - \sigma^2 = \frac{E[(Y - f_0)'] \left(M_{FM_{H_1'M_{H_2}}} V M_{FM_{H_1'M_{H_2}}} \right)^+ E(Y - f_0)}{n + q - k_1 - k_2}, \quad (48)$$

where

$$E(Y - f_0) = \frac{1}{2} \left[\kappa_{K_{H_1}\delta u} - FC_0^{-1}H_1'(H_1C_0^{-1}H_1' + H_2H_2')^{-1}\omega_{K_H\delta u} \right]$$

and $K_H = \begin{pmatrix} K_{H_1} \\ K_{H_2} \end{pmatrix}$ is a $(k_1 + k_2) \times (k_1 + k_2 - q)$ kernel matrix.

Proof. The expression (48) for the bias is a consequence of the properties of the probability distribution $\chi^2_f(\delta)$. The condition enables us to express the parameters $\delta\beta_1$ and $\delta\beta_2$ in the form

$$\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix} = \begin{pmatrix} K_{H_1} \\ K_{H_2} \end{pmatrix} \delta u - \frac{1}{2}(H_1, H_2)^- \omega_{K_H\delta u},$$

where one version of the g-inverse is

$$(H_1, H_2)^- = \begin{pmatrix} C_0^{-1}H_1'(H_1C_0^{-1}H_1' + H_2H_2')^{-1} \\ H_2'(H_1C_0^{-1}H_1' + H_2H_2')^{-1} \end{pmatrix} = \begin{pmatrix} T_0 \\ U_0 \end{pmatrix}$$

(for the analogy see Remark 2.3.3; it is to be noticed that the matrix $\begin{pmatrix} T_0 \\ U_0 \end{pmatrix}$ has the full rank in columns).

The new form of the model is

$$
\begin{aligned}
Y &= f_0 + FK_{H_1}\delta u + \frac{1}{2}\kappa_{K_{H_1}\delta u} - \frac{1}{2}FT_0\omega_{K_H\delta u} + \varepsilon \qquad (49)\\
\delta\beta_1 &= K_{H_1}\delta u - \frac{1}{2}T_0\omega_{K_H\delta u},\ \delta\beta_2 = K_{H_2}\delta u - \frac{1}{2}U_0\omega_{K_H\delta u},\ Var(Y) = \sigma^2 V.
\end{aligned}
$$

We can easily verify that $\mathcal{M}(K_{H_1}) = \mathcal{M}(M_{H_1' M_{H_2}})$ and $\mathcal{M}(K_{H_2}) = \mathcal{M}(M_{H_2' M_{H_1}})$, that implies that $\mathcal{M}(FK_{H_1}) = \mathcal{M}(FM_{H_1' M_{H_2}})$. Thus

$$
\begin{aligned}
&E[(Y-f_0)']\left(M_{FM_{H_1'M_{H_2}}} V M_{FM_{H_1'M_{H_2}}}\right)^+ E(Y-f_0) = \\
&= \frac{1}{4}(FK_{H_1}\delta u + \kappa_{K_{H_1}\delta u} - FT_0\omega_{K_H\delta u})'\left(M_{FM_{H_1'M_{H_2}}} V M_{FM_{H_1'M_{H_2}}}\right)^+ \times \\
&\quad \times (FK_{H_1}\delta u + \kappa_{K_{H_1}\delta u} - FT_0\omega_{K_H\delta u}) \\
&= \frac{1}{4}[\kappa_{K_{H_1}\delta u} - FC_0^{-1}H_1'(H_1C_0^{-1}H_1' + H_2H_2')^{-1}\omega_{K_H\delta u}]' \times \\
&\quad \times \left(M_{FM_{H_1'M_{H_2}}} V M_{FM_{H_1'M_{H_2}}}\right)^+ \times \\
&\quad \times [\kappa_{K_{H_1}\delta u} - FC_0^{-1}H_1'(H_1C_0^{-1}H_1' + H_2H_2')^{-1}\omega_{K_H\delta u}].
\end{aligned}
$$

♠

Definition 3.3.2 The symbol $K_{II,0}^{(int)}(\beta_0)$ in the model (49) has the following meaning

$$
\begin{aligned}
&K_{II,0}^{(int)}(\beta_0) = \qquad (50)\\
&= \sup\left\{ \frac{\sqrt{\boxed{5}'\left(M_{FM_{H_1'M_{H_2}}} V M_{FM_{H_1'M_{H_2}}}\right)^+ \boxed{5}}}{\delta s' K_{H_1}' F' V^{-1} F K_{H_1} \delta s} : \delta s \in R^{k_1+k_2-q} \right\}
\end{aligned}
$$

where

$$
\boxed{5} = \kappa_{K_{H_1}\delta s} - FT_0\omega_{K_H\delta s} = \kappa_{K_{H_1}\delta s} - FC_0^{-1}H_1'(H_1C_0^{-1}H_1' + H_2H_2')^{-1}\omega_{K_H\delta s}
$$

(cf. also Definition 2.3.9 for $\Sigma = V$).

Theorem 3.3.3 Let $\delta\beta_1 \in \mathcal{M}(K_{H_1}) = \mathcal{M}(M_{H_1' M_{H_2}})$. If

$$
\delta\beta_1' C_0 \delta\beta_1 \le \frac{2\varepsilon}{K_{II,0}^{(int)}(\beta_0)}\sqrt{n+q-k_1-k_2},
$$

then

$$
E(\hat{\sigma}^2) - \sigma^2 \le \varepsilon^2.
$$

Proof. Definition 3.3.2.and Lemma 3.3.1 imply

$$\begin{aligned} E(\hat{\sigma}^2) - \sigma^2 &= \frac{E[(Y - f_0)'] \left(M_{FM_{H_1'M_{H_2}}} V M_{FM_{H_1'M_{H_2}}} \right)^+ E(Y - f_0)}{n + q - k_1 - k_2} \\ &= \frac{1}{4(n + q - k_1 - k_2)} \boxed{5}' \left(M_{FM_{H_1'M_{H_2}}} V M_{FM_{H_1'M_{H_2}}} \right)^+ \boxed{5} \\ &\leq \left(K_{II,0}^{(int)}(\beta_0) \right)^2 \left(\frac{1}{2} \delta s' K_{H_1}' C_0 K_{H_1} \delta s \right)^2 \frac{1}{n + q - k_1 - k_2}. \end{aligned}$$

The further procedure is elementary. ♠

Remark 3.2.6 In all models considered $E(\hat{\sigma}^2) - \sigma^2 > 0$. Thus in the average the estimator is greater than the actual dispersion. Since the bias is of the 4th order in $\delta\beta$ it is not dangerous in the models with a weak nonlinearity.

4 The bias of estimators of variance components in mixed linear models

In this section the covariance matrix is assumed to be of the form

$$\Sigma(\vartheta) = \vartheta_1 V_1 + \ldots + \vartheta_p V_p,$$

where $\vartheta_i > 0,\ i = 1, \ldots, p$, and the matrices $V_1, \ldots V_p$ are positive semidefinite.

4.1 The model without constraints

Let

$$Y \sim N_n \left(f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta},\ \Sigma(\vartheta) \right), \tag{51}$$

$\delta\beta \in R^k,\ r(F) = k < n,\ \vartheta = (\vartheta_1, \ldots \vartheta_p)' \in \underline{\vartheta} \subset R^p$, where $\underline{\vartheta}$ is an open set with the property $\vartheta \in \underline{\vartheta} \Rightarrow \Sigma(\vartheta)$ is positive definite.

Let $\vartheta_0 \in \underline{\vartheta}$ be the approximate value of the actual value of the vector $\vartheta,\ \Sigma_0 = \Sigma(\vartheta_0)$. The matrix $S_{(M_F \Sigma_0 M_F)^+}$, where

$$\{S_{(M_F\Sigma_0 M_F)^+}\}_{i,j} = Tr[V_i(M_F\Sigma_0 M_F)^+ V_j (M_F \Sigma_0 M_H)^+],\ i, j = 1, \ldots, p,$$

is assumed to be regular.

Lemma 4.1.1 In the model $Y \sim N_n[f_0 + F\delta\beta,\ \Sigma(\vartheta)]$, $\delta\beta \in R^k$, the unbiased quadratic estimator with the minimum (in the Loewner sense) covariance matrix at the point ϑ_0 (in this case the ϑ_0-MINQUE) is

$$\hat{\vartheta} = S^{-1}_{(M_F\Sigma_0 M_F)^+} \begin{pmatrix} (Y-f_0)'(M_F\Sigma_0 M_F)^+ V_1 (M_F\Sigma_0 M_F)^+ (Y-f_0) \\ \vdots \\ (Y-f_0)'(M_F\Sigma_0 M_F)^+ V_p (M_F\Sigma_0 M_F)^+ (Y-f_0) \end{pmatrix}$$

and its covariance matrix at ϑ_0 is

$$Var(\hat{\vartheta}|\vartheta_0) = 2S^{-1}_{(M_F\Sigma_0 M_F)^+}.$$

Proof . Cf. [4]. ♠

Within the model (51) there holds

$$E(\hat{\vartheta}) - \vartheta = S^{-1}_{(M_F\Sigma_0 M_F)^+} \frac{1}{4} \begin{pmatrix} \kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ V_1 (M_F\Sigma_0 M_F)^+ \kappa_{\delta\beta} \\ \vdots \\ \kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ V_p (M_F\Sigma_0 M_F)^+ \kappa_{\delta\beta} \end{pmatrix}.$$

Theorem 4.1.2 If

$$\delta\beta' C_0 \delta\beta \le \frac{2\varepsilon}{K^{(int)}_{\Sigma_0}(\beta_0)},$$

where

$$K^{(int)}_{\Sigma_0}(\beta_0) = \sup\left\{ \frac{\sqrt{\kappa'_{\delta\beta}\Sigma_0^{-1} M_F^{\Sigma_0^{-1}} \kappa_{\delta\beta}}}{\delta\beta' F'\Sigma_0 F\delta\beta} : \delta\beta \in R^k \right\}, \tag{52}$$

then

$$\forall\{i = 1, \dots, p\} |E(\hat{\vartheta}_i) - \vartheta_i| \le \sum_{j=1}^{n} |k_{i,j}|\varepsilon^2;$$

here $k'_i = (k_{i,1}, \dots, k_{i,n}) = \left\{S^{-1}_{(M_F\Sigma_0 M_F)^+}\right\}_{i,\cdot} [Diag(\vartheta_0)]^{-1}$, $i = 1, \dots, p$.

Proof. Let

$$\hat{\zeta}_i = (Y-f_0)'(M_F\Sigma_0 M_F)^+ V_i (M_F\Sigma_0 M_F)^+ (Y-f_0);$$

then

$$E(\hat{\zeta}_i) = \left\{S_{(M_F\Sigma_0 M_F]^+}\right\}_{i,\cdot} \vartheta + \frac{1}{4}\kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ V_i (M_F\Sigma_0 M_F)^+ \kappa_{\delta\beta},$$

$i = 1 \dots, p$. If

$$\hat{\zeta} = (\hat{\zeta}_1, \dots, \hat{\zeta}_p)',$$

then

$$
\begin{aligned}
&E(\hat{\vartheta}) - \vartheta = \\
&= S^{-1}_{(M_F\Sigma_0 M_F)^+}\left[E(\hat{\zeta}) - S_{(M_F\Sigma_0 M_F)^+}\vartheta\right] \\
&= S^{-1}_{(M_F\Sigma_0 M_F)^+}\frac{1}{4}\begin{pmatrix} \kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ V_1 (M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} \\ \vdots \\ \kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ V_p (M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} \end{pmatrix} \\
&= S^{-1}_{(M_F\Sigma_0 M_F)^+}[Diag(\vartheta_0)]^{-1} \times \\
&\quad \times Diag(\vartheta_0)\frac{1}{4}\begin{pmatrix} \kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ V_1 (M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} \\ \vdots \\ \kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ V_p (M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} \end{pmatrix} \\
&= \begin{pmatrix} k'_1 \\ \vdots \\ k'_p \end{pmatrix}\frac{1}{4}\begin{pmatrix} \kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ \vartheta_{1,0} V_1 (M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} \\ \vdots \\ \kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ \vartheta_{p,0} V_p (M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} \end{pmatrix}.
\end{aligned}
$$

Since the matrices $V_1, \dots, V_p$ are assumed to be positive semidefinite and the parameters $\vartheta_{1,0}, \dots, \vartheta_{p,0}$ are positive, the inclusion $\mathcal{M}(V_i) \subset \mathcal{M}(\Sigma(\vartheta_0))$ is true, in a consequence of which

$$
\begin{aligned}
&\kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ \vartheta_{i,0} V_i (M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} \leq \\
&\leq \kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ \Sigma_0 (M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} = \kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+\kappa_{\delta\beta}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&|E(\hat{\vartheta}_i) - \vartheta_i| = \\
&= \frac{1}{4}\sum_{j=1}^{p} k_{i,j}\kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+ \vartheta_{j,0} V_j (M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} \\
&\leq \frac{1}{4}\sum_{j=1}^{p} |k_{i,j}|\kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+\kappa_{\delta\beta}.
\end{aligned}
$$

The definition of $K^{(int)}_{\Sigma_0}(\beta_0)$ (see (52)) implies

$$
\kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} \leq \left(K^{(int)}_{\Sigma_0}(\beta_0)\right)^2 (\delta\beta C_{\Sigma_0}\delta\beta)^2,
$$

where $C_{\Sigma_0} = F'\Sigma_0 F$. If

$$
\delta\beta' C_{\Sigma_0}\delta\beta \leq \frac{2\varepsilon}{K^{(int)}_{\Sigma_0}(\beta_0)},
$$

then $\kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} \leq 4\varepsilon^2$, thus also

$$E(\hat{\vartheta}_i) - \vartheta_i \leq \frac{1}{4}\sum_{j=1}^{p} |k_{i,j}|\kappa'_{\delta\beta}(M_F\Sigma_0 M_F)^+\kappa_{\delta\beta} \leq \sum_{j=1}^{p} |k_{i,j}|\varepsilon^2,$$

$i = 1, \ldots, p$. ♠

4.2 The model with constraints of the type I

Let

$$Y \sim N_n\left(f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta},\ \Sigma(\vartheta)\right),\ G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0,$$

$r(F) = k < n, r(G_{q,k}) = q < k,\ \vartheta \in \underline{\vartheta} \subset R^p$ and $\Sigma(\vartheta)$ is positive definite for $\vartheta \in \underline{\vartheta}$.

This model is equivalent with the model

$$Y \sim N_n\left(f_0 + FK_G\delta u + \frac{1}{2}(\kappa_{K_G\delta u} - FG^-_{m(C_{\Sigma_0})}\gamma_{K_G\delta u}),\ \Sigma(\vartheta)\right),\ \delta u \in R^{k-q},$$

where $C_{\Sigma_0} = F'\Sigma_0 F$ and K_G is a kernel matrix of G. In this model

$$K^{(int)}_{I,\Sigma_0}(\beta_0) = \sup\left\{\frac{\sqrt{\boxed{6}'(M_{FK_G}\Sigma_0 M_{FK_G})^+\boxed{6}}}{\delta u' K'_G C_{\Sigma_0} K_G \delta u} : \delta u \in R^{k-q}\right\}, \tag{53}$$

where $\boxed{6} = \kappa_{K_G\delta u} - FG^-_{m(C_{\Sigma_0})}\gamma_{K_G\delta u}$ and

$$\begin{aligned} E(\hat{\vartheta}_i) - \vartheta_i &= \\ &= k'_i\frac{1}{4}\begin{pmatrix} \boxed{6}'(M_{FK_G}\Sigma_0 M_{FK_G})^+V_1(M_{FK_G}\Sigma_0 M_{FK_G})^+\boxed{6} \\ \vdots \\ \boxed{6}'(M_{FK_G}\Sigma_0 M_{FK_G})^+V_p(M_{FK_G}\Sigma_0 M_{FK_G})^+\boxed{6} \end{pmatrix}, \end{aligned}$$

where

$$k'_i = \left\{S^{-1}_{(M_{FK_G}\Sigma_0 M_{FK_G})^+}\right\}_{i,\cdot}[Diag(\vartheta_0)]^{-1},\ i = 1, \ldots, p.$$

Theorem 4.2.1 If

$$\delta u' K'_G C_{\Sigma_0} K_G \delta u \leq \frac{2\varepsilon}{K^{(int)}_{I,\Sigma_0}(\beta_0)},$$

then

$$E(\hat{\vartheta}_i) - \vartheta_i \leq \sum_{j=1}^{p} |k_{i,j}|\varepsilon^2.$$

Proof . The assertion can be proved in an analogous way as Theorem 4.1.2. ♠

4.3 The model with constraints of the type II

Let

$$Y \sim N_n\left(f_0 + F\delta\beta_1 + \frac{1}{2}\kappa_{\delta\beta_1},\ \Sigma(\vartheta)\right), \delta\beta = \begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix},$$

$$H_1\delta\beta_1 + H_2\delta\beta_2 + \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} = 0$$

$r(F) = k_1 < n,\ r(H_1, H_2) = q < k_1 + k_2,\ r(H_2) = k_2 < q,\ \vartheta \in \underline{\vartheta} \subset R^p$, and $\Sigma(\vartheta)$ is positive definite for $\vartheta \in \underline{\vartheta}$.

This model is equivalent with the model

$$Y \sim N_n\left(f_0 + FK_{H_1}\delta u + \frac{1}{2}\kappa_{K_{H_1}\delta u} - FT_{\Sigma_0}\omega_{K_H\delta u}),\ \Sigma(\vartheta)\right),\ \delta u \in R^{k_1+k_2-q}, \tag{54}$$

where $K_H = \begin{pmatrix} K_{H_1} \\ K_{H_2} \end{pmatrix}$ is a kernel matrix of the matrix (H_1, H_2) and

$$\begin{pmatrix} T_{\Sigma_0} \\ U_{\Sigma_0} \end{pmatrix} = \begin{pmatrix} C_{\Sigma_0}^{-1}H_1'(H_1C_{\Sigma_0}^{-1}H_1' + H_2H_2')^{-1} \\ H_2'(H_1C_{\Sigma_0}^{-1}H_1' + H_2H_2')^{-1} \end{pmatrix}.$$

The parameters $\delta\beta_1$ and $\delta\beta_2$ can be expressed as follows

$$\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix} = \begin{pmatrix} K_{H_1} \\ K_{H_2} \end{pmatrix}\delta u - \frac{1}{2}\begin{pmatrix} T_{\Sigma_0} \\ U_{\Sigma_0} \end{pmatrix}\omega_{K_H\delta u}.$$

In the model (54)

$$K_{II,\Sigma_0}^{(int)}(\beta_0) = \sup\left\{\frac{\sqrt{\boxed{7}'(M_{FK_{H_1}}\Sigma_0 M_{FK_{H_1}})^+\boxed{7}}}{\delta u' K_{H_1}' F'\Sigma_0^{-1}FK_{H_1}\delta u} : \delta u \in R^{k_1+k_2-q}\right\}$$

where $\boxed{7} = \kappa_{K_{H_1}\delta u} - FT_{\Sigma_0}\omega_{K_H\delta u}$.

Theorem 4.3.1 If

$$\delta u' K_{H_1}' C_{\Sigma_0} K_{H_1}\delta u \leq \frac{2\varepsilon}{K_{II,\Sigma_0}^{(int)}(\beta_0)},$$

then

$$E(\hat{\vartheta}_i) - \vartheta_i \leq \sum_{j=1}^{p} |k_{i,j}|\varepsilon^2,$$

where

$$k_i' = (k_{i,1}, \ldots, k_{i,p}) = \left\{S_{(M_{FK_{H_1}}\Sigma_0 M_{FK_{H_1}})^+}^{-1}\right\}_{i.}[Diag(\vartheta_0)]^{-1},$$

$i = 1, \ldots, p$.

Proof. The assertion can be proved analogously as the assertion of Theorem 4.1.2. ♠

5 The confidence ellipsoid of a nonlinear vector function of the first order parameters

Let us consider a regular model

$$Y \sim N_n\left(f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta},\ \Sigma\right),\ \delta\beta \in R^k,$$

and an r-dimensional function $d(\cdot) : R^k \to R^r$ $(r < k)$ with continuous second derivatives. Its Taylor series at the point β_0 is

$$d(\beta) = d(\beta_0) + D\delta\beta + \frac{1}{2}\alpha_{\delta\beta},$$

where

$$\begin{aligned} D &= \partial d(\beta)/\partial\beta'|_{\beta=\beta_0},\ r(D) = r < k \\ \alpha_{\delta\beta} &= (\delta\beta' D_1 \delta\beta, \ldots, \delta\beta' D_r \delta\beta),\ D_i = \partial^2 d_i(\beta)/\partial\beta\partial\beta'|_{\beta=\beta_0}, \end{aligned}$$

$i = 1, \ldots, r$, $r(D_{r,k}) = r < k$.

Lemma 5.1. If $\delta\beta' C \delta\beta \leq \frac{2\varepsilon}{K^{(par)}(\beta_0)}$, then the ellipsoid

$$\mathcal{E}^* = \left\{u : u \in R^r, (u - D\delta\hat{\beta})'(DC^{-1}D')^{-1}(u - D\delta\hat{\beta}) \leq \left(\sqrt{\chi^2_r(0; 1-\alpha)} + \varepsilon\right)^2\right\}$$

covers the vector $D\delta\beta$ with the probability at least $1 - \alpha - \eta$, where

$$P\{\chi^2_r(\varepsilon^2) \leq \chi^2_r(0; 1-\alpha)\} = 1 - \alpha - \eta.$$

Proof. Let $\delta\beta' C \delta\beta \leq \frac{2\varepsilon}{K^{(par)}(\beta_0)}$. Then (cf. Corollary 2.1.4) for $h = D'u$ and $c_b\sqrt{\chi^2_r(0; 1-\alpha)} = \varepsilon$,

$$\forall\{u \in R^r\} |u' Db| \leq \varepsilon\sqrt{u' DC^{-1}D'u}$$

(here $b = E(\delta\hat{\beta}) - \delta\beta$), that according to the Scheffé theorem is equivalent with the inequality

$$b'D'(DC^{-1}D')^{-1}Db \leq \varepsilon^2.$$

The bias Db of the estimator $D\delta\hat{\beta}$ is inside the ellipsoid

$$\{u : u'(DC^{-1}D')^{-1}u \leq \varepsilon^2\}, \tag{55}$$

which is homothetical with the confidence ellipsoid in the linearized model.

Let the spectral decomposition of the matrix $(DC^{-1}D')^{-1}$ be

$$(DC^{-1}D')^{-1} = \sum_{i=1}^{r} \lambda_i f_i f_i'.$$

The confidence ellipsoid in the linearized model is

$$\{u : u \in R^r, (u - D\delta\hat{\beta})'(DC^{-1}D')^{-1}(u - D\delta\hat{\beta}) \leq \chi_r^2(0; 1-\alpha)\}$$

and its semiaxes a_i are $a_i = \sqrt{\chi_r^2(0; 1-\alpha)}/\sqrt{\lambda_i}$, $i = 1, \dots, r$. The semiaxes of the ellipsoid (55) are $\overline{a}_i = \varepsilon/\sqrt{\lambda_i}$, $i = 1, \dots, r$. The ellipsoid $\mathcal{E}^*$ has the semiaxes $a_i + \overline{a}_i$, $i = 1, \dots, r$, and covers all those confidence ellipsoids in the linearized model whose centers deflect from the vector $D\delta\hat{\beta}$ by the vectors Db lying in the ellipsoid (55).

The random variable

$$(D\delta\beta - D\delta\hat{\beta})'(DC^{-1}D')^{-1}(D\delta\beta - D\delta\hat{\beta})$$

has $\chi_r^2(\delta)$ probability distribution with the parameter of noncentrality

$$\delta = (Db)'(DC^{-1}D')^{-1}Db < \varepsilon^2.$$

If η fulfils the relation

$$P\{\chi_r^2(\varepsilon^2) \leq \chi_r^2(0; 1-\alpha)\} = 1 - \alpha - \eta,$$

then the ellipsoid $\mathcal{E}^*$ covers the vector $D\delta\beta$ with the probability greater or equalling $1 - \alpha - \eta$. ♠

In the following the notation

$$C_d(\beta_0) = \\ = \sup\left\{ \frac{\sqrt{(\alpha_{\delta\beta} - DC^{-1}F'\Sigma^{-1}\kappa_{\delta\beta})'(DC^{-1}D')^{-1}(\alpha_{\delta\beta} - DC^{-1}F'\Sigma^{-1}\kappa_{\delta\beta})}}{\delta\beta' C \delta\beta} \right. \\ \left. : \delta\beta \in R^k \right\}.$$

Theorem 5.2 Consider the model $Y \sim N_n\left(f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta},\ \Sigma\right)$ and the function $d(\cdot) : R^k \to R^r$. If

$$\delta\beta' C \delta\beta \leq \frac{2\varepsilon}{C_d(\beta_0)},$$

then the ellipsoid

$$\begin{aligned}\mathcal{E}^* &= \Big\{u : u \in R^r, (u - D\delta\hat{\beta})'(DC^{-1}D')^{-1}(u - D\delta\hat{\beta}) \\ &\leq \left(\sqrt{\chi_r^2(0; 1-\alpha)} + \varepsilon\right)^2\Big\} + d(\beta_0) + D\delta\hat{\beta}\end{aligned}$$

covers the value $d(\beta)$ with the probability greater or equalling $1 - \alpha - \eta$, where

$$P\{\chi_r^2(\varepsilon^2) \leq \chi_r^2(0; 1-\alpha)\} = 1 - \alpha - \eta.$$

Proof. The random variable

$$[d(\beta) - d(\beta_0) - D\delta\hat{\beta}]'(DC^{-1}D')^{-1}[d(\beta) - d(\beta_0) - D\delta\hat{\beta}]$$

has the $\chi_r^2(\delta)$ probability distribution with the parameter of noncentrality

$$\delta = \xi'(DC^{-1}D')^{-1}\xi,$$

where

$$\begin{aligned}\xi &= d(\beta) - d(\beta_0) - DE(\delta\hat{\beta}) \\ &= d(\beta) - d(\beta_0) - D\delta\beta + Db \\ &= \frac{1}{2}(\alpha_{\delta\beta} - DC^{-1}F'\Sigma^{-1}\kappa_{\delta\beta}).\end{aligned}$$

If $\widehat{d(\beta)} = d(\beta_0) + D\delta\hat{\beta}$, then $\xi = E[\widehat{d(\beta)}] - d(\beta)$.

The definition of $C_d(\beta_0)$ implies

$$\xi'(DC^{-1}D')^{-1}\xi \leq C_d^2(\beta_0)\left(\frac{1}{2}\delta\beta'C\delta\beta\right)^2.$$

If $\delta\beta'C\delta\beta \leq 2\varepsilon/C_d(\beta_0)$, then $\xi'(DC^{-1}D')^{-1}\xi \leq \varepsilon^2$ and the vector $\xi = E[\widehat{d(\beta)}] - d(\beta)$ is an element of the ellipsoid

$$\{u : u \in R^r,\ u'(DC^{-1}D')^{-1}u \leq \varepsilon^2\}.$$

The further procedure is analogous to that of the proof of the preceding lemma. ♠

The following two statements (see [7, pp. 53-54]) allowe us to determine the confidence region of the parameter β without assumptions on a weak nonlinearity of the regression model.

Let us consider an r-dimensional function $d(\cdot) : R^k \to R^r$, given at the beginning of the section. In the following two lemmas the notation $D(\beta) = \partial d(\beta)/\partial\beta$ is used.

Lemma 5.3 Let $Y \sim N_n[d(\beta), \Sigma]$. Then the set

$$\left\{\beta : [y - d(\beta)]' \left(P_{D(\beta)}^{\Sigma^{-1}}\right)' \Sigma^{-1} P_{D(\beta)}^{\Sigma^{-1}} [y - d(\beta)] \leq \chi_k^2(0; 1-\alpha)\right\}$$

covers the actual value of the parameter β with the probability $1 - \alpha$.

Lemma 5.4 Let $Y \sim N_n[d(\beta), \sigma^2 V]$, where σ^2 is unknown. Then the set

$$\left\{\beta : \frac{(n-k)A(\beta)}{kB(\beta)} \leq F_{k,n-k}(0; 1-\alpha)\right\},$$

where

$$\begin{aligned} A(\beta) &= [y - d(\beta)]' \left(P_{D(\beta)}^{V^{-1}}\right)' V^{-1} P_{D(\beta)}^{V^{-1}} [y - d(\beta)], \\ B(\beta) &= [y - d(\beta)]' \left(M_{D(\beta)}^{V^{-1}}\right)' V^{-1} M_{D(\beta)}^{V^{-1}} [y - d(\beta)], \end{aligned}$$

covers the actual value of the parameter β with the probability $1 - \alpha$.

6 The bias of a function of an estimated parameter

Let us have a function $p(\cdot) : R^k \to R^r$ with continuous second derivatives. Let the vector $p(\beta)$ be estimated by the statistic $\widehat{p(\beta)} = p(\beta_0) + J\delta\hat{\beta}$, where $J = \partial p(u)/\partial u'|_{u=\beta_0}$. Then

1. within the model $Y \sim_n \left(f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \Sigma\right)$

   $$\delta\hat{\beta} = C^{-1}F'\Sigma^{-1}(Y - f_0)$$

   and

   $$E[\widehat{p(\beta)}] - p(\beta) = JC^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{\delta\beta} - \frac{1}{2}\begin{pmatrix} \delta\beta' P_1 \delta\beta \\ \vdots \\ \delta\beta' P_r \delta\beta \end{pmatrix},$$

   where $P_i = \partial^2 p_i(u)/\partial u \partial u'|_{u=\beta_0}$, $i = 1, \dots, r$;

2. within the model $Y \sim_n \left(f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}, \Sigma\right), G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0$,

   $$\delta\hat{\beta} = M_{C^{-1}G'}^{C} C^{-1}F'\Sigma^{-1}(y - f_0),$$

   and

   $$\begin{aligned} E[\widehat{p(\beta)}] - p(\beta) &= JC^{-1}G'(GC^{-1}G')^{-1}\frac{1}{2}\gamma_{K_G\delta u} + \\ &+ J[I - C^{-1}G'(GC^{-1}G')^{-1}G]C^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{K_G\delta u} - \\ &- \frac{1}{2}\begin{pmatrix} \delta u' K_G' P_1 K_G \delta u \\ \vdots \\ \delta u' K_G' P_r K_G \delta u \end{pmatrix}. \end{aligned}$$

3. Within the model

$$Y \sim_n \left(f_0 + F\delta\beta_1 + \frac{1}{2}\kappa_{\delta\beta_1},\ \Sigma\right),\ H_1\delta\beta_1 + H_2\delta\beta_2 + \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} = 0,$$

let us consider a function $p(\cdot,\cdot\cdot) : R^{k_1+k_2} \to R^r$, with continuous second derivatives. If the vector $p(\beta_1,\beta_2)$ is estimated by the statistic

$$\widehat{p(\beta_1,\beta_2)} = p(\beta_{1,0},\beta_{2,0}) + (J_1,J_2)\begin{pmatrix}\delta\hat{\beta}_1\\ \delta\hat{\beta}_2\end{pmatrix},$$

where

$$J_1 = \frac{\partial p(u,v)}{\partial u'}\Big|_{\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}\beta_{1,0}\\ \beta_{2,0}\end{pmatrix}},\ J_2 = \frac{\partial p(u,v)}{\partial v'}\Big|_{\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}\beta_{1,0}\\ \beta_{2,0}\end{pmatrix}}$$

and

$$\begin{pmatrix}\delta\hat{\beta}_1\\ \delta\hat{\beta}_2\end{pmatrix} =$$
$$= \begin{pmatrix} I \\ -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1\end{pmatrix}\left(M_{H_1'M_{H_2}}CM_{H_1'M_{H_2}}\right)^+ F'\Sigma^{-1}(y-f_0),$$

then

$$E[\widehat{p(\beta_1,\beta_2)}] - p(\beta_1,\beta_2) =$$
$$= (J_1,J_2)\left[\begin{pmatrix} I \\ -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1\end{pmatrix} M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{K_{H_1}\delta u}\right.$$
$$\left. + \begin{pmatrix} -C^{-1}H_1(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+ \\ ((H_2')^-_{m(H_1C^{-1}H_1')})'\end{pmatrix}\frac{1}{2}\omega_{K_H\delta u}\right] -$$
$$-\frac{1}{2}\begin{pmatrix} \delta u'(K_{H_1}', K_{H_2}')\begin{pmatrix}P_{1,11}, & P_{1,12}\\ P_{1,21}, & P_{1,22}\end{pmatrix}\begin{pmatrix}K_{H_1}\\ K_{H_2}\end{pmatrix}\delta u \\ \vdots \\ \delta u'(K_{H_1}', K_{H_2}')\begin{pmatrix}P_{r,11}, & P_{r,12}\\ P_{r,21}, & P_{r,22}\end{pmatrix}\begin{pmatrix}K_{H_1}\\ K_{H_2}\end{pmatrix}\delta u\end{pmatrix},$$

where
$$P_{i,jk} = \frac{\partial^2 p_i(u_j,u_k)}{\partial u_j \partial u_k'}\Big|_{u=\beta_0},\quad u = \begin{pmatrix}u_1\\ u_2\end{pmatrix}, \beta_0 = \begin{pmatrix}\beta_{1,0}\\ \beta_{2,0}\end{pmatrix},$$

$i = 1,\dots,r; j,k = 1,2.$

Remark 6.1 The linearization region in all three cases can be determined using the following procedure. Let us express the vector $E[\widehat{p(\beta)}] - p(\beta)$ in the form

$$E[\widehat{p(\beta)}] - p(\beta) = \frac{1}{2}\begin{pmatrix} \delta\beta' A_1 \delta\beta \\ \vdots \\ \delta\beta' A_r \delta\beta \end{pmatrix};$$

e.g. in the case 1

$$A_i = \left(\sum_{j=1}^{r} \{JC^{-1}F'\Sigma^{-1}\}_{i,j} \frac{\partial^2 p_j(u)}{\partial u \partial u'}\Big|_{u=\beta_0} - P_i\right), i = 1, \ldots, r.$$

Let $h \in R^r$ be an arbitrary vector and $\frac{1}{2}A_h = \frac{1}{2}\sum_{i=1}^{r} h_1 A_i$. Let $\frac{1}{2}A_h = \sum_{i=1}^{r} \lambda_i c_i c_i'$ be the spectral decomposition of the matrix $\frac{1}{2}A_h$. Let us denote $\frac{1}{2}\overline{A}_h = \sum_{i=1}^{r} |\lambda_i| c_i c_i'$. Then obviously

$$\forall \left\{\delta\beta : \frac{1}{2}\delta\beta' \overline{A}_h \delta\beta \le \varepsilon\right\} |h'\{E[\widehat{p(\beta)}] - p(\beta)\}| \le \varepsilon.$$

Let

$$E[\widehat{p(\beta)}] - p(\beta) = \frac{1}{2}\begin{pmatrix} \delta\beta' A_1 \delta\beta \\ \vdots \\ \delta\beta' A_r \delta\beta \end{pmatrix} = \frac{1}{2}\alpha_{\delta\beta}$$

and

$$K_p(\beta_0) = \sup\left\{\frac{\sqrt{\alpha'_{\delta\beta}[JVar(\delta\hat{\beta})J']^{-1}\alpha_{\delta\beta}}}{\delta\beta'[Var(\delta\hat{\beta})]^{-1}\delta\beta} : \delta\beta \in R^k\right\}.$$

Theorem 6.2 If

$$\delta\beta'[Var(\delta\hat{\beta})]^{-1}\delta\beta \le \frac{2\varepsilon}{K_p(\beta_0)},$$

then

$$\forall\{h \in R^r\}|h'\alpha_{\delta\beta}| \le \varepsilon\sqrt{h'JVar(\delta\hat{\beta})J'h}.$$

Proof. Obviously

$$\left|h'\frac{1}{2}\alpha_{\delta\beta}\right| \le \sqrt{h'JVar(\delta\hat{\beta})J'h}\sqrt{\frac{1}{4}\alpha'_{\delta\beta}[JVar(\delta\hat{\beta})J']^{-1}\alpha_{\delta\beta}}.$$

The further procedure of proving is elementary. ♠

7 Influence of the choice of β_0 on $Var(\delta\hat{\beta})$

In the regular model $Y \sim_n \left(f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta},\ \Sigma\right)$, $\delta\beta \in R^k$, the covariance matrix of the estimator $\delta\hat{\beta}$ is

$$Var(\delta\hat{\beta}) = (F'\Sigma^{-1}F)^{-1} = C^{-1}.$$

A great distance between the actual value and the chosen approximate value β_0 that influences the matrix F can cause a nonnegligible disturbance of the matrix C^{-1}. In the following a criterion is given which determines the bounderies of the admissible shift $\delta\beta$.

Lemma 7.1 Let C be a $k \times k$ positive definite matrix and E a $k \times k$ symmetric matrix. Let $C = Q\Delta Q'$ be the spectral decomposition of the matrix C and $\Delta^{-1/2}Q'EQ\Delta^{-1/2} = FDF'$ the spectral decomposition of the matrix $\Delta^{-1/2}Q'EQ\Delta^{-1/2}$. Then the matrix $R = F'\Delta^{-1/2}Q'$ is a matrix with the properties

$$RCR' = I \quad \text{and} \quad RER' = D = Diag(d_1, \ldots, d_k).$$

Proof. If $C = Q\Delta Q'$ is the spectral decomposition of the matrix C, then $Q'CQ = \Delta$ and $\Delta^{-1/2}Q'CQ\Delta^{-1/2} = I$. Further if $\Delta^{-1/2}Q'EQ\Delta^{-1/2} = FDF'$ is the spectral decomposition of the matrix $\Delta^{-1/2}Q'EQ\Delta^{-1/2}$ then $F'\Delta^{-1/2}Q'EQ\Delta^{-1/2}F = D$ and simultaneously $F'\Delta^{-1/2}Q'CQ\Delta^{-1/2}F = I.$♠

Criterion 7.2 Let C be a $k \times k$ positive definite matrix and E a $k \times k$ symmetric matrix. The matrix E is said to fulfil Criterion 7.2 if

$$|d_i| < 1,$$

$i = 1, \ldots, k$ where $D = Diag(d_1, \ldots, d_k)$, see Lemma 7.1.

Lemma 7.3 Let C be a $k \times k$ positive definite matrix and E an $k \times k$ symmetric matrix such that they fulfil Criterion 7.2. Then

$$(C + E)^{-1} = C^{-1} - C^{-1}EC^{-1} + C^{-1}EC^{-1}EC^{-1} - \ldots.$$

Proof. For the matrices C and E there exists a regular matrix R of the type $k \times k$ such that

$$RCR' = I \quad \text{and} \quad RER' = D = Diag(d_1, \ldots, d_k),$$

see Theorem 7.1.

Thus

$$[R(C + E)R']^{-1} = (I + D)^{-1} = Diag\left(\frac{1}{1 + d_1}, \ldots, \frac{1}{1 + d_k}\right).$$

Since the matrix E is assumed to fulfil Criterion 7.2, according to which $|d_i| < 1$, each element of the diagonal matrix $(I + D)^{-1}$ can be expressed in the absolutely convergent series

$$\frac{1}{1 + d_i} = 1 - d_i + d_i^2 - d_i^3 + \dots .$$

This implies

$$[R(C + E)R']^{-1} = (I + D)^{-1} = I - D + D^2 - D^3 + \dots ,$$

that is equivalent with

$$\begin{aligned}
&(C + E)^{-1} = \\
&= R'R - R'RR^{-1}D(R')^{-1}R'R + R'RR^{-1}D(R')^{-1}R'RR^{-1}D(R')^{-1}R'R + \dots = \\
&= C^{-1} - C^{-1}EC^{-1} - C^{-1}EC^{-1}EC^{-1} + \dots ,
\end{aligned}$$

since $R'R = C^{-1}$ and $R^{-1}D(R')^{-1} = E$. ♠

In the following we denote

$$\Delta = \begin{pmatrix} \delta\beta' H_1 \\ \vdots \\ \delta\beta' H_n \end{pmatrix}.$$

The shift $\delta\beta$ from the point β_0 causes that the matrix C changes to the matrix

$$C + \Delta'\Sigma^{-1}F + F'\Sigma^{-1}\Delta.$$

Let the shift be assumed to be so small, that the matrix $\Delta'\Sigma^{-1}F + F'\Sigma^{-1}\Delta$ fulfils Criterion 7.2.

Then the error in the dispersion $Var(h'\delta\hat{\beta}) = h'C^{-1}h$ caused by the mentioned shift in β is

$$h'(C + \Delta'\Sigma^{-1}F + F'\Sigma^{-1}\Delta)^{-1}h - h'C^{-1}h,$$

and this, using Lemma 7.3, can be expressed in the form

$$h'C^{-1}(\Delta'\Sigma^{-1}F + F'\Sigma^{-1}\Delta)C^{-1}h.$$

Therefore it is natural to define the linearization region as the set

$$\left\{\delta\beta : \frac{|h'C^{-1}(\Delta'\Sigma^{-1}F + F'\Sigma^{-1}\Delta)C^{-1}h|}{h'C^{-1}h} \leq \varepsilon\right\}.$$

As far as the expression $h'C^{-1}(\Delta'\Sigma^{-1}F + F'\Sigma^{-1}\Delta)C^{-1}h$ is concerned, it holds

$$h'C^{-1}(\Delta'\Sigma^{-1}F + F'\Sigma^{-1}\Delta)C^{-1}h = \delta\beta'\alpha_h, \tag{56}$$

where

$$\alpha_h = (h'A_1h, \dots, h'A_kh)',$$

$$A_i = C^{-1}(H_i^*)'\Sigma^{-1}FC^{-1} + C^{-1}F'\Sigma^{-1}H_i^*C^{-1},\ i = 1, \dots, k,$$

and

$$H_i^* = \begin{pmatrix} e_i'H_1 \\ \vdots \\ e_i'H_n \end{pmatrix},\ i = 1, \dots, k.$$

Definition 7.4

$$C^{(Var)}(\beta_0) = \sup\left\{\frac{\sqrt{\alpha_h'C^{-1}\alpha_h}}{h'C^{-1}h} : h \in R^k\right\}. \tag{57}$$

Theorem 7.5 If $\delta\beta'C\delta\beta \le \frac{\varepsilon}{C^{(Var)}(\beta_0)}$, then

$$\forall\{h \in R^k\}\frac{|h'C^{-1}(\Delta'\Sigma^{-1}F + F'\Sigma^{-1}\Delta)C^{-1}h|}{h'C^{-1}h} \le \varepsilon.$$

Proof. According to (56) and the Schwarz inequality

$$|h'C^{-1}(\Delta'\Sigma^{-1}F + F'\Sigma^{-1}\Delta)C^{-1}h| = |\delta\beta'\alpha_h| \le \sqrt{\delta\beta'C\delta\beta}C^{(Var)}(\beta_0)h'C^{-1}h.$$

The rest of the proof is elementary. ♠

8 Nonlinearity and hypotheses testing

8.1 The model without constraints

In the model $Y \sim N_n\left(f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta},\ \Sigma\right)$, $\delta\beta \in R^k$, the null hypothesis

$$H_0 : t(\beta) = 0,$$

is tested against the alternative

$$H_a : t(\beta) \neq 0.$$

In the following the function $t(\cdot) : R^k \to R^r$ is considered in the form $t(\beta) = t(\beta_0) + T\delta\beta + \frac{1}{2}\tau_{\delta\beta}$, where

$$T = \frac{\partial t(u)}{\partial u'}\Big|_{u=\beta_0},\quad \tau_{\delta\beta} = (\delta\beta'T_1\delta\beta, \dots, \delta\beta T_r\delta\beta)' \quad\text{and}\quad T_i = \frac{\partial^2 t_i(u)}{\partial u\partial u'}\Big|_{u=\beta_0},$$

$i = 1, \ldots, r$; the matrix T is of full rank in the rows.

In the following $t(\beta_0) = 0$ is supposed.

Let the test statistic be $(T\delta\hat{\beta})'(TC^{-1}T')^{-1}T\delta\hat{\beta}$. Its probability distribution is $\chi^2_r(\delta)$, where

$$\delta = [TE(\delta\hat{\beta})]'(TC^{-1}T')^{-1}TE(\delta\hat{\beta}).$$

If the null hypothesis is true, then $T\delta\beta + \frac{1}{2}\tau_{\delta\beta} = 0$. Let K_T be a $k \times (k-r)$ matrix with the property $\mathcal{M}(K_T) = \mathcal{K}er(T)$. Thus if H_0 is true, $\delta\beta$ can be expressed in the form $\delta\beta = K_T\delta u - T^-_{m(C)}\frac{1}{2}\tau_{K_T\delta u}$ (the terms of the higher than second order are neglected) and

$$\begin{aligned} TE(\delta\hat{\beta}) &= E[TC^{-1}F'\Sigma^{-1}(Y - f_0)] = TC^{-1}F'\Sigma^{-1}\left(F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}\right) = \\ &= TC^{-1}F'\Sigma^{-1}\left[F\left(K_T\delta u - T^-_{m(C)}\frac{1}{2}\tau_{K_T\delta u}\right) + \frac{1}{2}\kappa_{K_T\delta u}\right] = \\ &= -\frac{1}{2}\tau_{K_T\delta u} + TC^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{K_T\delta u} = \frac{1}{2}\boxed{\mathrm{t}}. \end{aligned}$$

Thus if the null hypothesis is true, the noncentrality parameter δ_0 is

$$\delta_0 = \frac{1}{4}\boxed{\mathrm{t}}'(TC^{-1}T')^{-1}\boxed{\mathrm{t}}.$$

Analogously to $K_I^{(constr)}(\beta_0)$ let us introduce $K^{(test)}(\beta_0)$.

Definition 8.1.1

$$K^{(test)}(\beta_0) = \sup\left\{\frac{\sqrt{\boxed{\mathrm{t}}'(TC^{-1}T')^{-1}\boxed{\mathrm{t}}}}{\delta u' K_T' C K_T \delta u} : \delta u \in R^{k-r}\right\}.$$

Theorem 8.1.2 If

$$\delta u' K_T' C K_T \delta u \leq \frac{2\sqrt{\delta_{\max}}}{K^{(test)}(\beta_0)},$$

where

$$P\{\chi^2_r(\delta_{\max}) \geq \chi^2_r(0; 1-\alpha)\} = \alpha + \eta,$$

then $\delta_0 \leq \delta_{\max}$ and the risk of the test caused by the nonlinearity is smaller than $\alpha + \eta$.

Proof. If the null hypothesis is true,

$$\delta_0 = \frac{1}{4}\boxed{\mathrm{t}}'(TC^{-1}T')^{-1}\boxed{\mathrm{t}} \leq \left(K^{(test)}(\beta_0)\right)^2 \frac{1}{4}(\delta u' K_T' C K_T \delta u)^2.$$

If $\delta u' K_T C K_T \delta u$ does not exceed the value $2\sqrt{\delta_{\max}}/K^{(test)}(\beta_0)$, then $\delta_0 \leq \delta_{\max}$. ♠

Remark 8.1.3 The course of the power function in the nonlinear model is shifted to the higher values and therefore the nonlinearity is dangerous only for the risk. Since the increase of the power function in the alternative hypothesis is favourable, the case of the null hypothesis was studied only.

8.2 The model with constraints of the type I

In the model $Y \sim N_n\left(f_0 + F\delta\beta + \frac{1}{2}\kappa_{\delta\beta},\ \Sigma\right)$, $G\delta\beta + \frac{1}{2}\gamma_{\delta\beta} = 0$, the null hypothesis

$$H_0:\ t(\beta) = 0,$$

is tested against the alternative

$$H_a:\ t(\beta) \neq 0.$$

The function $t(\cdot) : R^k \to R^r$ is considered in the form $t(\beta) = T\delta\beta + \frac{1}{2}\tau_{\delta\beta}$, where T is a matrix of the type $r \times k$ whose rank is $r(T) = r < k$. Furthermore $r\begin{pmatrix} G \\ T \end{pmatrix} = r + q < k$.

Lemma 8.2.1 Let W be a positive semidefinite $n \times n$ matrix, $r(W) = r$ and K_W an $n \times (n-r)$ matrix such that $\mathcal{K}er(W) = \mathcal{M}(K_W)$. If the matrix B of the type $s \times n$ has the rank $r(B) = s$ and simultaneously $\mathcal{M}(B') \cap \mathcal{M}(K_W) = \{0\}$, then BWB' is positive definite.

Proof. Let K_W has the property $K_W' K_W = I$. Then the spectral decomposition of the matrix W can be written in the form

$$W = (F, K_W)\begin{pmatrix} \Lambda, & 0 \\ 0, & 0 \end{pmatrix}\begin{pmatrix} F' \\ K_W' \end{pmatrix},$$

where $\Lambda = Diag(\lambda_1, \ldots, \lambda_r), \lambda_i > 0, i = 1, \ldots, r$, and thus

$$\begin{pmatrix} F' \\ K_W' \end{pmatrix} W(F, K_W) = \begin{pmatrix} \Lambda, & 0 \\ 0, & 0 \end{pmatrix}.$$

Each matrix B' of full rank in columns such that $\mathcal{M}(B') \cap \mathcal{M}(K_W) = \{0\}$, can be written in the form

$$B' = (F, K_W)\begin{pmatrix} U \\ V \end{pmatrix},$$

where U is of full rank in columns. Thus

$$BWB' =$$
$$= (U',V')\begin{pmatrix} F' \\ K'_W \end{pmatrix}(F,K_W)\begin{pmatrix} \Lambda, & 0 \\ 0, & 0 \end{pmatrix}\begin{pmatrix} F' \\ K'_W \end{pmatrix}(F,K_W)\begin{pmatrix} U \\ V \end{pmatrix} = U'\Lambda U,$$

since (F, K_W) is an orthogonal matrix. The matrix $U'\Lambda U$ is obviously positive definite. ♠

Under our assumption the matrix $T(M_{G'}CM_{G'})^{+}T'$ is regular, since

$$\mathcal{K}er[(M_{G'}CM_{G'})^{+}] = \mathcal{M}(G') \quad \text{and} \quad \mathcal{M}(T') \cap \mathcal{M}(G') = \{0\}.$$

For testing the null hypothesis $H_0 : T\delta\beta + \frac{1}{2}\tau_{\delta\beta} = 0$ the test statistic

$$(T\delta\hat{\hat{\beta}})'[T(M_{G'}CM_{G'})^{+}T']^{-1}T\delta\hat{\hat{\beta}},$$

where $\delta\hat{\hat{\beta}} = M^{C}_{C^{-1}G'}C^{-1}F'\Sigma^{-1}(Y - f_0)$, will be used. This statistic has $\chi^2_r(\delta)$ probability distribution, where the noncentrality parameter is

$$\delta = [TE(\delta\hat{\hat{\beta}})]'[T(M_{G'}CM_{G'})^{+}T']^{-1}TE(\delta\hat{\hat{\beta}}).$$

For $TE(\delta\hat{\hat{\beta}}) = TE[M^{C}_{C^{-1}G'}C^{-1}F'\Sigma^{-1}F(Y - f_0)]$ we obtain

$$\begin{aligned} TE(\delta\hat{\hat{\beta}}) &= TM^{C}_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\left(F\delta\beta + \frac{1}{2}\kappa_{\delta\beta}\right) = \\ &= TM^{C}_{C^{-1}G'}\left(\delta\beta + C^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{\delta\beta}\right) = \\ &= T(I - P^{C}_{C^{-1}G'})\delta\beta + TM^{C}_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{\delta\beta} = \\ &= T\delta\beta - TP^{C}_{C^{-1}G'}\delta\beta + TM^{C}_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{\delta\beta}. \end{aligned}$$

If H_0 is true, then $T\delta\beta = -\frac{1}{2}\tau_{\delta\beta}$ and furthermore

$$\begin{aligned} -TP^{C}_{C^{-1}G'}\delta\beta &= -TC^{-1}G'(GC^{-1}G')^{-1}G\delta\beta = \\ &= TC^{-1}G'(GC^{-1}G')^{-1}\frac{1}{2}\gamma_{\delta\beta}. \end{aligned}$$

Thus

$$\begin{aligned} &TE(\delta\hat{\hat{\beta}}) = \\ &= -\frac{1}{2}\tau_{\delta\beta} + TC^{-1}G'(GC^{-1}G')^{-1}\frac{1}{2}\gamma_{\delta\beta} + TM^{C}_{C^{-1}G'}C^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{\delta\beta} = \\ &= \frac{1}{2}\boxed{\text{t}}_{I}. \end{aligned}$$

Analogously as in the previous subsection let us define $K_I^{(test)}(\beta_0)$.

Definition 8.2.2

$$K_I^{(test)}(\beta_0) = \sup\left\{ \frac{\sqrt{\boxed{t}'_I[T(M_{G'}CM_{G'})^+T']^{-1}\boxed{t}_I}}{\delta s' K'_{(G',T')'}(M_{G'}CM_{G'})^+K_{(G',T')'}\delta s} : \delta s \in R^{k-(q+r)} \right\}.$$

Here $K_{(G',T')'}$ is a $k \times (k-q-r)$-matrix with the property

$$\mathcal{K}er\begin{pmatrix} G \\ T \end{pmatrix} = \mathcal{M}\left(K_{(G',T')'}\right).$$

Obviously

$$\mathcal{M}\left(K_{(G',T')'}\right) = \mathcal{M}(K_G) \cap \mathcal{M}(K_T),$$

where $\mathcal{K}er(G) = \mathcal{M}(K_G)$ and $\mathcal{K}er(T) = \mathcal{M}(K_T)$.

Lemma 8.2.3

$$\begin{pmatrix} G \\ T \end{pmatrix}^-_{m(C)} = \left(P^{M_{G'}CM_{G'}}_{\mathcal{K}er(T)} G^-_{m(C)}, P^{M_{T'}CMT'}_{\mathcal{K}er(G)} T^-_{m(C)}\right).$$

Proof. It is necessary to check that

$$\begin{pmatrix} G \\ T \end{pmatrix}\begin{pmatrix} G \\ T \end{pmatrix}^-_{m(C)}\begin{pmatrix} G \\ T \end{pmatrix} = \begin{pmatrix} G \\ T \end{pmatrix}$$

and

$$C\begin{pmatrix} G \\ T \end{pmatrix}^-_{m(C)}\begin{pmatrix} G \\ T \end{pmatrix} = (G',T')\left[\begin{pmatrix} G \\ T \end{pmatrix}^-_{m(C)}\right]' C.$$

It is simple however tedious and therefore it is ommitted. ♠

Since

$$\mathcal{K}er\begin{pmatrix} G \\ T \end{pmatrix} = \mathcal{M}\left[I - \begin{pmatrix} G \\ T \end{pmatrix}^- \begin{pmatrix} G \\ T \end{pmatrix}\right],$$

Lemma 8.2.3 can be used for determining the explicite form of the matrix generating $\mathcal{K}er\begin{pmatrix} G \\ T \end{pmatrix}$.

The matrix $K_{(G',T')}$ enables us to express the linear terms of the vector $\delta\beta$ that fulfils the null hypothesis, i.e.

$$\delta\beta = K_{(G',T')}\delta u - \begin{pmatrix} G \\ T \end{pmatrix}^-_{m(C)} \frac{1}{2}\begin{pmatrix} \gamma \\ \tau \end{pmatrix}_{K_{(G',T')}\delta u}.$$

Theorem 8.2.4 If

$$\delta u' K'_{(G',T')'} (M_{G'} C M_{G'})^{+} K_{(G',T')'} \delta u \leq \frac{2\sqrt{\delta_{\max}}}{K_I^{(test)}(\beta_0)},$$

where

$$P\{\chi_r^2(\delta_{\max}) \geq \chi_r^2(0; 1-\alpha)\} = \alpha + \eta,$$

then the noncentrality parameter δ is smaller than $\delta_{\max}$ and therefore the risk of the test is not greater than $\alpha + \eta$.

Proof is analogous to the proof of Theorem 8.1.2. ♠

8.3 The model with constraints of the type II

In the model

$$Y \sim N_n\left(f_0 + F\delta\beta_1 + \frac{1}{2}\kappa_{\delta\beta_1}, \Sigma\right), \; H_1\delta\beta_1 + H_2\delta\beta_2 + \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} = 0,$$

the null hypothesis

$$H_0 : t(\beta_1, \beta_2) = 0,$$

is tested against the alternative

$$H_a : t(\beta_1, \beta_2) \neq 0.$$

Here $t(\beta_1, \beta_2) = T_1\delta\beta_1 + T_2\delta\beta_2 + \frac{1}{2}\tau_{(\delta\beta_1,\delta\beta_2)}$, where the $r \times k_1$ matrix T_1 and the $r \times k_2$ matrix T_2 fulfil the condition

$$r\begin{pmatrix} H_1, & H_2 \\ T_1, & T_2 \end{pmatrix} = r + q.$$

The test statistic is of the form

$$\left[(T_1, T_2)\begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix}\right]' \left[(T_1, T_2) Var\begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix}\begin{pmatrix} T_1' \\ T_2' \end{pmatrix}\right]^{-1} (T_1, T_2)\begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix},$$

where

$$\begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix} = \begin{pmatrix} I \\ -[(H_2')^{-}_{m(H_1C^{-1}H_1')}]'H_1 \end{pmatrix} M^{C}_{C^{-1}H_1'M_{H_2}} C^{-1}F'\Sigma^{-1}(Y - f_0)$$

and

$$Var\begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \left(M_{H_1'M_{H_2}} C M_{H_1'M_{H_2}}\right)^+, & -C^{-1}H_1'(H_2')^-_{m(H_1C^{-1}H_1')} \\ -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1C^{-1}, & [(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1C^{-1}H_1'(H_2')^-_{m(H_1C^{-1}H_1')} \end{pmatrix}.$$

The test statistic has the $\chi^2_r(\delta)$ probability distribution with the noncentrality parameter

$$\delta = \left[(T_1,T_2)E\begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix}\right]' \left[(T_1,T_2)Var\begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix}\begin{pmatrix} T_1' \\ T_2' \end{pmatrix}\right]^{-1} (T_1,T_2)E\begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix}.$$

Here

$$\begin{aligned}
&(T_1,T_2)E\begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix} = \\
&= (T_1,T_2)\begin{pmatrix} I \\ -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1 \end{pmatrix} \times \\
&\quad \times \left(\left(I - P^C_{C^{-1}H_1'M_{H_2}}\right)\delta\beta_1 + M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{\delta\beta_1}\right) \\
&= (T_1,T_2)\begin{pmatrix} \delta\beta_1 \\ -[(H_2')^-_{m(H_1C^{-1}H_1')}]'\left(-H_2\delta\beta_2 - \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)}\right) \end{pmatrix} - \\
&\quad -(T_1,T_2)\begin{pmatrix} I \\ -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1 \end{pmatrix} P^C_{C^{-1}H_1'M_{H_2}}\delta\beta_1 + \\
&\quad +(T_1,T_2)\begin{pmatrix} I \\ -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1 \end{pmatrix} M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{\delta\beta_1}.
\end{aligned}$$

Further the equalities

$$\begin{aligned}
&P^C_{C^{-1}H_1'M_{H_2}}\delta\beta_1 = \\
&= C^{-1}H_1'(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+H_1\delta\beta_1 = \\
&= C^{-1}H_1'(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+\left(-H_2\delta\beta_2 - \frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)}\right) = \\
&= -C^{-1}H_1'(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+\frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)}
\end{aligned}$$

and

$$T_1\delta\beta_1 + T_2\delta\beta_2 = -\frac{1}{2}\tau_{(\delta\beta_1,\delta\beta_2)}$$

(it holds if the null hypothesis H_0 is true) are used. They imply

$$
\begin{aligned}
&(T_1,T_2)E\begin{pmatrix}\delta\hat{\hat{\beta}}_1\\ \delta\hat{\hat{\beta}}_2\end{pmatrix} = \\
&= -\frac{1}{2}\tau_{(\delta\beta_1,\delta\beta_2)} + T_2[(H_2')^-_{m(H_1C^{-1}H_1')}]'\frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} + \\
&\quad +(T_1,T_2)\begin{pmatrix} I\\ -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1\end{pmatrix}C^{-1}H_1'(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+\frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} + \\
&\quad +(T_1,T_2)\begin{pmatrix} I\\ -[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_1\end{pmatrix}M^C_{C^{-1}H_1'M_{H_2}}C^{-1}F'\Sigma^{-1}\frac{1}{2}\kappa_{\delta\beta_1}.
\end{aligned}
$$

Taking into account the relation

$$
\begin{aligned}
&T_2[(H_2')^-_{m(H_1C^{-1}H_1')}]'\left[\frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} - H_1C^{-1}H_1'(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+\frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)}\right] = \\
&= T_2[(H_2')^-_{m(H_1C^{-1}H_1')}]'H_2[(H_2')^-_{m(H_1C^{-1}H_1')}]'\frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)} = \\
&= T_2[(H_2')^-_{M(H_1C^{-1}H_1')}]'\frac{1}{2}\omega_{(\delta\beta_1,\delta\beta_2)},
\end{aligned}
$$

we get

$$
(T_1,T_2)E\begin{pmatrix}\delta\hat{\hat{\beta}}_1\\ \delta\hat{\hat{\beta}}_2\end{pmatrix} = \frac{1}{2}\boxed{t}_{II},
$$

where

$$
\begin{aligned}
\boxed{t}_{II} &= -\tau_{(\delta\beta_1,\delta\beta_2)} + (T_1,T_2)\left[\begin{pmatrix} M^C_{C^{-1}H_1'M_{H_2}}\\ -\left((H_2')^-_{m(H_1C^{-1}H_1')}\right)'H_1\end{pmatrix}C^{-1}F'\Sigma^{-1}\kappa_{\delta\beta_1}+\right.\\
&\left.+\begin{pmatrix} C^{-1}H_1'(M_{H_2}H_1C^{-1}H_1'M_{H_2})^+\\ \left((H_2')^-_{m(H_1C^{-1}H_1')}\right)'\end{pmatrix}\omega_{(\delta\beta_1,\delta\beta_2)}\right].
\end{aligned}
$$

The symbol $K = \begin{pmatrix}K_1\\ K_2\end{pmatrix}$ means a matrix of the type $(k_1+k_2)\times(k_1+k_2-q-r)$ such that

$$
\mathcal{K}er\begin{pmatrix}H_1, & H_2\\ T_1, & T_2\end{pmatrix} = \mathcal{M}\begin{pmatrix}K_1\\ K_2\end{pmatrix}
$$

$$
\left(\Rightarrow \mathcal{M}\begin{pmatrix}K_1\\ K_2\end{pmatrix} = \mathcal{K}er(H_1,H_2)\cap\mathcal{K}er(T_1,T_2)\right).
$$

If the null hypothesis H_0 is true, the parameter $\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix}$ can be expressed as

$$\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \delta u + \text{ terms of the second order}, \quad \delta u \in R^{k_1+k_2-q-r}.$$

Definition 8.3.1

$$K_{II}^{(test)}(\beta_0) =$$

$$= \sup \left\{ \frac{\sqrt{\boxed{\text{t}}'_{II} \left[(T_1, T_2) Var \begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix} \begin{pmatrix} T'_1 \\ T'_2 \end{pmatrix} \right]^{-1} \boxed{\text{t}}_{II}}}{\delta u'(K'_1, K'_2) \left[Var \begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix} \right]^{+} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \delta u} : \delta u \in R^{k_1+k_2-q-r} \right\},$$

where $\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \delta u$ in the expression for $\boxed{\text{t}}_{II}$.

With respect to Lemma 8.2.1 the matrix

$$(T_1, T_2) Var \begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix} \begin{pmatrix} T'_1 \\ T'_2 \end{pmatrix}$$

is regular and since

$$\mathcal{M} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \subset \mathcal{M} \left[Var \begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix} \right],$$

the matrix

$$(K'_1, K'_2) \left[Var \begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix} \right]^{+} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

is invariant to the choice of the g-inverse and regular.

Theorem 8.3.2 If $\begin{pmatrix} \delta\beta_1 \\ \delta\beta_2 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \delta u$ fulfils the inequality

$$\delta u'(K'_1, K'_2) \left[Var \begin{pmatrix} \delta\hat{\hat{\beta}}_1 \\ \delta\hat{\hat{\beta}}_2 \end{pmatrix} \right]^{+} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \delta u \le \frac{2\sqrt{\delta_{\max}}}{K_{II}^{(test)}(\beta_0)},$$

where $\delta_{\max}$ fulfils the equation

$$P\{\chi^2_r(\delta_{\max}) \ge \chi^2_r(0; 1-\alpha)\} = \alpha + \eta,$$

then the parameter of noncentrality δ (in the case that the null hypothesis is true) does not exceed the value $\delta_{\max}$ and therefore the risk is not greater than $\alpha + \eta$.

Proof is an analogy to the proof of Theorem 8.1.2. ♠

Remark 8.3.3 The explicite expression of the matrix $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ is not found yet. The numerical solution is elementary, it suffices the apply the relation

$$\mathcal{K}er \begin{pmatrix} H_1, & H_2 \\ T_1, & T_2 \end{pmatrix} = \mathcal{M}\left[\begin{pmatrix} I, & 0 \\ 0, & I \end{pmatrix} - \begin{pmatrix} H_1, & H_2 \\ T_1, & T_2 \end{pmatrix}^{-} \begin{pmatrix} H_1, & H_2 \\ T_1, & T_2 \end{pmatrix}\right].$$

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