New Mathematical Self-calibration Models in Aerial Photogrammetry

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Summary: A new family of self-calibration additional parameters (APs) is presented for calibrating airborne camera systems. We point out that photogrammetric self-calibration can to a very large extent be considered as a function approximation problem in mathematics. Based on the mathematical approximation theory, a novel family of so-called Legendre APs is developed based on the orthogonal Legendre polynomials. It is permitted in theory that Legendre APs can fully calibrate the image distortion of all digital frame-format airborne cameras, as long as the proper degree of APs has been chosen. It is also shown, that Legendre APs can be considered as the superior generalization of the conventional polynomials APs proposed by Ebner and Grün in the 1970s. The good performance of Legendre APs is demonstrated by using the data from the recent DGPF project and from other field tests. Despite the promising results, however, we finally reveal an essential deficiency of all polynomials APs, caused by the high correlations. Algebraic polynomials may be not the optimal mathematical choice and further work is desired for more rigorous self-calibration.


1 Introduction

Aerial photogrammetry has seen a great revolution in the last decades. The navigation system, typically GPS/IMU integration, has been successfully incorporated into photogrammetry. It leads to direct georeferencing on camera exterior orientation and so-called integrated sensor orientation. This technique reduces the number of the ground control points (GCPs), increases realibility and flexibility and accelerates the photogrammetric mapping. Moreover, the introduction of various digital airborne cameras has greatly advanced the field. Most of the digital frame airborne cameras utilize multi-head construction and virtual large-image composition, while the latest DMC II cameras use very large single head system. Along with these remarkable developments, challenges have been raised. Of them, camera calibration is still a most crucial topic.

Camera calibration is an essential subject in photogrammetry and computer vision. Self-calibration by using additional parameters (APs) has been widely accepted and substantially utilized as an efficient calibration technique in photogrammetric society since 1970s. Traditionally, two types of self-calibration APs were developed for analogue single-head camera calibration: physical and
mathematical. The development of physical APs was mainly attributed to D. C. Brown (Brown, 1971) for close-range camera calibration and the APs later extended by attaching additional polynomials for aerial application (Brown, 1976). The mathematical APs (or ‘polynomials APs’) were proposed by Ebner (1976) and Grun (1978), by using two- and four-order orthogonal bivariate polynomials respectively. For a review on the early work on camera calibration, see Clarke & Fryer (1998). These APs, though being widely used for many years and even in digital era, may not be capable to fit the distinctive features of digital cameras, such as multi-head and virtual composition.

A considerable progress was made for the digital camera calibration. Cramer (2009) and Jacobsen et al. (2010) reported comprehensive empirical tests, in which lots of different APs were employed to compensate the image distortion. However, many of them are purely the combinations of the traditional APs while lack of solid physical or mathematical foundations. Further, the extra systematic effect caused by direct georeferencing demand overall system calibration rather than lens distortion calibration only (Honkavaara, 2004, Cramer et al., 2010). Thus, self-calibration APs must be decoupled from the other correction parameters.

All the above motivate our present work on airborne camera calibration. We review the mathematical principle of self-calibration and point out that self-calibration can to a very large extent be considered as a function approximation problem in mathematics. Based on the mathematical approximation theory, a novel family of so-called Legendre APs is developed. The Legendre APs are orthogonal, rigorous, flexible and effective. The high performance of Legendre APs is extensively evaluated by using empirical test data. We also compare Legendre APs with other traditional APs from both theoretical and practical viewpoints. It is shown that Legendre APs can be considered in many ways as the superior generalization of the polynomials APs presented by Ebner (1976) and Grun (1978).

In spite of the encouraging results delivered by Legendre APs, we reveal one intrinsic deficiency of all polynomials APs, including those of Ebner (1976), Grun (1978) and Legendre APs. This deficiency is mainly caused by high correlations of their parameters. Consequently, polynomials may be a natural but probably not the optimal choice for self-calibration.

While some supplementary materials of our study on Legendre APs were presented in Tang et al. (2012), we will emphasize the mathematical principles behind self-calibration APs in this paper. In Section 2, the theory of polynomials approximation is reviewed and Legendre APs are constructed. The practical tests are demonstrated in Section 3. The comparisons are made between the Legendre APs and the traditional counterparts in Section 4, as well as the mathematical criticisms on polynomials APs.

2 Legendre self-calibration APs

The collinearity equation which is the mathematical fundamental of photogrammetry reads as follows.

\[
\begin{align*}
\Delta x &= x_0 - c_1 \left( X - X_0 \right) + c_2 \left( Y - Y_0 \right) + c_3 \left( Z - Z_0 \right) + \Delta x(x, y) + \epsilon \\
\Delta y &= y_0 - c_4 \left( X - X_0 \right) + c_5 \left( Y - Y_0 \right) + c_6 \left( Z - Z_0 \right) + \Delta y(x, y) + \epsilon
\end{align*}
\]

(1)

where \( \Delta x \) and \( \Delta y \) denote image distortion, \( \epsilon \) random error. The denotations of other parameters can be seen in textbooks such like Kraus (2007). The image distortion terms, \( \Delta x(x, y) \) and \( \Delta y(x, y) \), are two-variable functions whose form is unknown. They have to be approximated by some models, i.e., self-calibration APs.

This means, we need to seek a parametric model which can represent accurately the actual lens distortion. Do note that there are several kinds of basis functions available in mathematics, whose combinations can approximate any unknown functions. Therefore, the distortion can be approximated by the linear combinations of the specific basis functions, even if little is known on the distortion. The coefficients of linear combinations can be fixed during the adjustment process.
Therefore, photogrammetric self-calibration can to a very large extent be considered as a function approximation or, more precisely, a curve fitting problem in mathematics. In this paper, we adopt the algebraic polynomials as the basis functions for self-calibration. The general principle of the polynomial approximation is outlined as follows.

2.1 Orthogonal polynomials approximation

The algebraic polynomial approximation is founded on the Weierstrass Theorem (seen in many textbooks such as MASON & HANDSCOMB, 2002). It indicates that any univariate function can be approximated with arbitrary accuracy by a polynomial of sufficiently high degree. Among all the possible forms, the orthogonal polynomials (OPs) are often favored in both theoretical and practical applications. The OPs can be categorized into two types: discrete and continuous. The former is orthogonal on specific discrete measurements while the later is orthogonal over the whole domain of definition.

For the curve fitting problem, the analytical form of the function is unknown while some sample measurements are available. The unknown function can be approximated by the combination of OPs. If the number of the measurements is close to the degree of the used polynomials, the discrete OPs are usually employed and can be obtained by orthogonalization process. It is noteworthy that the discrete OPs are orthogonal on the measured locations only, but not necessarily on the others. Else, if the measurements are very dense and the number is much larger than the polynomials’ degree, the continuous OPs is preferred. More theoretical materials can be seen in such as BERZTIS (1964) and MASON & HANDSCOMB (2002).

Legendre Polynomials, denoted by \( \{ L_m(x) \}_{m=0,1,...} \) where \( m \) indicates the order, are a series of continuous OPs over \([-1,1]\), i.e.,

\[
L_m(x) \leq 1, \quad -1 \leq x \leq 1
\]  
(2)

\[
\int_{-1}^{1} L_m(x)L_n(x)dx = \begin{cases} 
0, & m \neq n \\
1, & m = n 
\end{cases}
\]  
(3)

Legendre polynomials grant the optimal approximation in the least-square sense (MASON & HANDSCOMB, 2002) and are widely used in many applications. The first few normalized Legendre Polynomials are listed in appendix A.

Legendre polynomials are complete bivariate OPs over the rectangular domain \([-1,1] \times [-1,1]\), satisfying

\[
\int_{-1}^{1} \int_{-1}^{1} p_{m,n}(x,y)p_{i,j}(x,y)dxdy = \begin{cases} 
1, & \text{if } m = i \text{ and } n = j \\
0, & \text{else}
\end{cases}
\]  
(5)

“Complete” indicates that any two-variable function can be approximated well by the \( \{ p_{m,n} \} \)

(KOORNWINDER, 1975).

2.2 Self-calibration APs

Let \( 2b_x \) and \( 2b_y \) denote the width and length of the image format, respectively. By scaling we obtain,
\[
l_m(x, b_i) = L_m(x/b_i) \quad (6)
\]
\[
l_n(y, b_j) = L_n(y/b_j) \quad (7)
\]

where \( x \) and \( y \) are the metric image coordinates, \( L_m \) and \( L_n \) are univariate Legendre Polynomials \((m, n = 0, 1, 2, \ldots)\). The first few \( \{l_m(x, b_i)\}_m \) are,

\[
\begin{align*}
l_0(x, b_i) &= 1 \\
l_1(x, b_i) &= x/b_i \\
l_2(x, b_i) &= \left(3\left(x/b_i\right)^2 - 1\right)/2 \\
l_3(x, b_i) &= \left(5\left(x/b_i\right)^2 - 3\left(x/b_i\right)\right)/2 \\
l_4(x, b_i) &= \left(35\left(x/b_i\right)^4 - 30\left(x/b_i\right)^2 + 3\right)/8 \\
l_5(x, b_i) &= \left(63\left(x/b_i\right)^5 - 70\left(x/b_i\right)^3 + 15\left(x/b_i\right)\right)/8 \\
l_6(x, b_i) &= \left(231\left(x/b_i\right)^6 - 315\left(x/b_i\right)^4 + 105\left(x/b_i\right)^2 - 5\right)/16
\end{align*}
\]

Similar formulae of \( \{l_n(y, b_j)\}_n \) could be derived. Denote

\[
f_{m,n} \triangleq f_{m,n}(x, y; b_i, b_j) = l_m(x, b_i)l_n(y, b_j) \quad (8)
\]

then \( \{f_{m,n}\}_{m,n} \) are the bivariate OPs over the rectangular frame \([ -b_i, b_i ] \times [ -b_j, b_j ] \) and \( |f_{m,n}| \leq 1 \). Considering the image distortion is typically in the order of \( \mu \mu \), we obtain \( p_{m,n} \) by multiplying \( f_{m,n} \) with \( 10^{-n} \) for numerical stability.

\[
p_{m,n} = 10^{-n} f_{m,n}, \quad |p_{m,n}| \leq 10^{-n} \quad (9)
\]

\( \{p_{m,n}\}_{m,n} \) can be ordered lexicographically as Eq. (10), following KOORNWINDER (1975).

\[
p_{0,0} \\
p_{1,0}, p_{0,1} \\
p_{2,0}, p_{1,1}, p_{0,2} \\
p_{3,0}, p_{2,1}, p_{1,2}, p_{0,3} \\
p_{4,0}, p_{3,1}, p_{2,2}, p_{1,3}, p_{0,4} \\
\ldots .
\]

Obviously,

\[
\int_{-b}^{b} \int_{-b}^{b} p_{m,n} \, dxdy = 0 \quad \text{if} \quad i \neq m \quad \text{or} \quad j \neq n \quad (11)
\]
It indicates that if the image measurements are densely distributed, then

$$\sum_i p_{i,0}(x_i, y_i)p_{n,0}(x_i, y_i) \approx 0 \text{ if } i \neq m \text{ or } j \neq n$$ \hspace{1cm} (12)

Eq. (12) implies that $\{p_{n,0}\}_{n=0}^{\infty}$ is (almost) orthogonal over the all image measurements.

Therefore, the bivariate distortion $\Delta x(X, Y)$ and $\Delta y(X, Y)$ in Eq. (1) could be repetively approximated by a series of continuous OPs $\{p_{i,n}\}_{n=0}^{\infty}$ and $\{p_{i,n}\}_{n=0}^{\infty}$, where $M_x$ and $N_y$ are the chosen maximum degrees which are not necessarily equal. Further, six of them should be eliminated, as done by EiNNER (1976) and GRUN (1978). Specially, the constant terms $p_{0,0}$ in $\Delta x(X, Y)$ and $\Delta y(X, Y)$ are nothing but the principle point offset; $p_{0,0}$, $p_{0,1}$, $p_{2,0}$ and $p_{1,1}$ in $\Delta x(X, Y)$ are highly correlated with $p_{0,1}$, $p_{1,0}$, $p_{2,0}$ and $p_{0,2}$ in $\Delta y(X, Y)$, respectively. Thus, the number of the unknown parameters is $(M_x + 1)(N_y + 1) + (M_x + 1)(N_x + 1) - 6$.

As examples, the APs with $M_x = M_y = 5$ and $N_x = N_y = 5$ are obtained in Eq. (13) with 66 unknown parameters ($a_{i,j}, i = 1, 2, \ldots, 66$).

$$\Delta x = a_{1,0} p_{1,0} + a_{1,1} p_{1,1} + a_{1,3} p_{1,3} + a_{1,5} p_{1,5} + a_{1,7} p_{1,7} + a_{2,0} p_{2,0} + a_{2,1} p_{2,1} + a_{2,3} p_{2,3} + a_{2,5} p_{2,5} + a_{2,7} p_{2,7} + a_{3,0} p_{3,0} + a_{3,1} p_{3,1} + a_{3,3} p_{3,3} + a_{3,5} p_{3,5} + a_{3,7} p_{3,7} + a_{4,0} p_{4,0} + a_{4,1} p_{4,1} + a_{4,3} p_{4,3} + a_{4,5} p_{4,5} + a_{4,7} p_{4,7} + a_{5,0} p_{5,0} + a_{5,1} p_{5,1} + a_{5,3} p_{5,3} + a_{5,5} p_{5,5} + a_{5,7} p_{5,7}$$

$$\Delta y = a_{2,0} p_{2,0} + a_{2,1} p_{2,1} + a_{2,3} p_{2,3} + a_{2,5} p_{2,5} + a_{2,7} p_{2,7} + a_{3,0} p_{3,0} + a_{3,1} p_{3,1} + a_{3,3} p_{3,3} + a_{3,5} p_{3,5} + a_{3,7} p_{3,7} + a_{4,0} p_{4,0} + a_{4,1} p_{4,1} + a_{4,3} p_{4,3} + a_{4,5} p_{4,5} + a_{4,7} p_{4,7} + a_{5,0} p_{5,0} + a_{5,1} p_{5,1} + a_{5,3} p_{5,3} + a_{5,5} p_{5,5} + a_{5,7} p_{5,7}$$

(13)

The APs with 34 unknowns, $M_x = M_y = 4$ and $N_x = N_y = 3$, are given in Eq. (14).

$$\Delta x = a_{1,0} p_{1,0} + a_{1,1} p_{1,1} + a_{1,3} p_{1,3} + a_{1,5} p_{1,5} + a_{1,7} p_{1,7} + a_{2,0} p_{2,0} + a_{2,1} p_{2,1} + a_{2,3} p_{2,3} + a_{2,5} p_{2,5} + a_{2,7} p_{2,7} + a_{3,0} p_{3,0} + a_{3,1} p_{3,1} + a_{3,3} p_{3,3} + a_{3,5} p_{3,5} + a_{3,7} p_{3,7} + a_{4,0} p_{4,0} + a_{4,1} p_{4,1} + a_{4,3} p_{4,3} + a_{4,5} p_{4,5} + a_{4,7} p_{4,7} + a_{5,0} p_{5,0} + a_{5,1} p_{5,1} + a_{5,3} p_{5,3} + a_{5,5} p_{5,5} + a_{5,7} p_{5,7}$$

$$\Delta y = a_{2,0} p_{2,0} + a_{2,1} p_{2,1} + a_{2,3} p_{2,3} + a_{2,5} p_{2,5} + a_{2,7} p_{2,7} + a_{3,0} p_{3,0} + a_{3,1} p_{3,1} + a_{3,3} p_{3,3} + a_{3,5} p_{3,5} + a_{3,7} p_{3,7} + a_{4,0} p_{4,0} + a_{4,1} p_{4,1} + a_{4,3} p_{4,3} + a_{4,5} p_{4,5} + a_{4,7} p_{4,7} + a_{5,0} p_{5,0} + a_{5,1} p_{5,1} + a_{5,3} p_{5,3} + a_{5,5} p_{5,5} + a_{5,7} p_{5,7}$$

(14)

So far the whole family of APs has been completely constructed. The input of APs includes the image length and width ($2b_x$ and $2b_y$), and the chosen degrees ($M_x$, $N_x$, $M_y$, and $N_y$). Usually, it can further adopt $M_x = M_y = \tilde{M}$ and $N_x = N_y = \tilde{N}$ in practice. This class of APs is based on
Legendre Polynomials and thus called Legendre APs.

2.3 Overall system calibration

As mentioned previously, the systematic effects caused by direct georeferencing must be compensated. The effects of important interest include the misalignment between the camera and the navigation instruments and the shift/drift in direct georeferencing, if present. For the overall system calibration, one of the most challenging works is to minimize the coupling effect of the different correction parameters. The decoupling is of vital importance in the sense that each systematic error must be independently and properly corrected and the calibration results are block-invariant.

For this purpose, we suggest the joint application of the Legendre APs (for calibrating the image distortion) with other correction parameters, i.e., the three interior orientation (IO) parameters used for correcting the principle point offset and the focal length deformation, and GPS/IMU shift/drift and misalignment correction parameters. The low correlation must be warranted among these calibration parameters and between them and exterior orientation (EO). As will be seen in Section 4, the correlations between Legendre APs and EO, and between Legendre APs and other correlation parameters, are fairly small. The low correlation is one advantage of Legendre APs over the traditional APs.

3 Practical tests

The Legendre APs are tested by using the data from the recent DGPF camera calibration project (German Society for Photogrammetry, Remote Sensing and Geoinformation), which was performed under the umbrella of DGPF and carried out in the test field Vaihingen/Enz nearby Stuttgart, Germany (KRÄMER, 2010, DGPF website, 2010). This successful project aims at an independent and comprehensive evaluation on the performance of digital airborne cameras, as well as offering a standard empirical dataset for the next years.

Four flights’ data of the frame cameras are adopted: DMC (GSD 20cm, ground sample distance), DMC (GSD 8cm), UltracamX (GSD 20cm) and UltracamX (GSD 8cm). Each camera is flown at two heights. For each flight, we are interested in two most often contexts: the in-situ calibration one and the operational project one. The former context is with high side overlapping (~60%) and dense GCPs and the later with low side overlapping (~20%) and few GCPs. The block configuration is depicted in Appendix C and the readers are referred to the relevant references for the project details.

3.1 In-situ calibration context

The system calibration strategy in Section 2.3 is adopted for all the blocks. Particularly, IMU misalignment, horizontal GPS shift (factually insignificant in our tests), IO parameters and Legendre APs with $M_i = N_x = M_j = N_y = 5$ are employed. The order of Legendre APs is empirically selected by the compromise between achieving the optimal accuracy and reducing overparameterization. The derived external accuracy, indicated by “self calibrating”, would be compared to the theoretical accuracy and the “without APs” one, for which the same correction parameters except Legendre APs are used.

The derived external accuracy is demonstrated in Fig. 1. By comparing “Self calibrating” with “Without APs”, the refinement of Legendre APs is significant in all tests, up to 10 cm in the DMC (GSD 20cm) block. Moreover, all the “self calibrating” accuracy reaches very close to the theoretical one and it means that the optimal accuracy has been achieved. All the “self calibrating” accuracy reaches to 1/5 GSD in the horizontal directions and 2/5 GSD in the vertical directions in four blocks. It is also interesting to notice that although the DMC and UltracamX cameras are differently manufactured, very similar external accuracy can be obtained by using Legendre APs in the blocks of similar configuration, i.e., similar GSD, similar forward and side overlapping levels and similar GCPs distribution. This fact, independent of the used cameras, coincides well with our photogrammetric accuracy expectation.

Now look at the estimation of the precision of the image measurements. The posterior std. dev.
estimation is 1.6, 1.4, 0.89 and 0.78 (unit: \( \mu m \)) for DMC (GSD 20cm, GSD 8cm) and UltracamX (GSD 20cm, GSD 8cm) blocks, respectively. These values are around 0.12 pixel size, which are 12 \( \mu m \) and 7.2 \( \mu m \) for DMC and UltracamX cameras, respectively. They well match the expected precision of the automatic tie point transfer techniques, which are 0.1-0.2 pixel size for aerial images.

**Fig. 1:** External accuracy in four in-situ calibration blocks, dense GCPs and p60%-q60% (‘without APs’ indicates without using Legendre APs only)

### 3.2 Operational project context

There are 4 GCPs and 20% side overlapping level in each block, which is much weaker than the in-situ calibration context. The IMU misalignment, IO parameters and the Legendre APs with \( M_x = M_y = 4, N_x = N_y = 3 \) are employed in the adjustment. Using Legendre APs of lower degree tries to avoid the potential overparameterization. This derived external accuracy is analogously denoted as “self calibrating” one. Due to 4 GCPs available only, the GPS/IMU observations have to be weighted carefully to achieve best accuracy.

We also evaluate the quality of the in-situ calibration in last sub-section. The calibration results of IO parameters and image distortion in Section 3.1 are utilized as fixed known values in the adjustment of the corresponding “reduced” operational block, i.e., the cameras are assumed being calibrated and need no further self-calibration. The derived external accuracy is named as “after calibration”. We compare “after calibration” with “self calibrating”, “without APs” and theoretical ones.

The adjust accuracy in four blocks is illustrated in Fig. 2. From those results, the self-calibrating Legendre APs help improve the external accuracy and the “after calibration” yields further refinement, more than 1/2 GSD in DMC (GSD 8cm) block. It is also interesting to see that for the blocks of p60%-q20% and few GCPs, the accuracy in vertical direction is general worse than 1/2 GSD. Nevertheless, the “self-calibrating” accuracy can be even worse for larger blocks. The “after calibration” accuracy is very close to the optimal theoretical one in every block. Therefore, these tests not only recognize the sufficient accuracy obtained by Legendre APs in the operational projects, but also confirm again their great efficiency in the in-situ calibration.

It is also worth mentioning that the Legendre APs have also been assessed by the flight data of other airborne cameras in other test fields, like medium-format DigiCAM and large-format UltracamXp. The similarly good results are confirmed while the details are not published here.
Fig. 2: External accuracy in four operational project blocks, 4 GCPs and p60%-q20% (’without APs’ indicates without using Legendre APs only).

4 Discussions

4.1 Comparisons

In this section, we make comparisons between Legendre APs and the conventional APs from the theoretical and practical viewpoints.

First, the mathematical links between Legendre APs and Ebner and Grün APs can be derived as follows. Assume that there are \((2m+1)\times(2m+1)\) points distributed equidistantly on a square image dimension. We consider the two-degree monic polynomial \(p_1(x)\) (the leading coefficient is one). It corresponds to the term \(k\) in Ebner and Grün APs (see Appendix B) and \(p_{2,0}\) in Legendre APs, respectively. Neglecting one scale factor \((b=1\text{ for Ebner and Grün APs, see Appendix B})\), \(p_2(x)\) could be obtained as Eq. (15) by using the Gram-Schmidt process.

\[
p_2(x) = x^2 - \frac{\sum_{j=-m}^{m} \sum_{j=-m}^{m} (j/m)^2}{\sum_{j=-m}^{m} \sum_{j=-m}^{m} 1} = x^2 - \frac{(2m+1) \sum_{j=-m}^{m} (j/m)^2}{(2m+1)^2} = x^2 - \frac{m+1}{3m}
\]  

(15)

Then,

\[
p_2(x) = x^2 - 2/3 = k, \text{ if } m = 1 \text{ (3×3 point distribution, Ebner APs)}
\]

\[
p_2(x) = x^2 - 1/2 = k, \text{ if } m = 2 \text{ (5×5 point distribution, Grün APs)}
\]

\[
p_1(x) = x^2 - 1/3 = p_{2,0}, \text{ if } m \rightarrow \infty \text{ (Legendre APs)}
\]

This is the exact relation between the discrete and the continuous orthogonality, as mentioned in Section 2.1. Therefore, the polynomials used in Ebner and Grün APs are belong to discrete orthogonal polynomials while Legendre APs are continuous orthogonal polynomials. In fact, Ebner and Grün
APs can also be derived in the exact way as Legendre APs. Briefly speaking, orthogonalizing univariate polynomials as Eq. 15 (over 3 and 5 equidistant points); then applying Eq. (4) and eliminating six parameters will obtain Ebner and Grün APs. Ebner and Grün restricted their APs in the assumed regular 3×3 and 5×5 “grid points” configurations. It is known, somewhat confusingly, that both Ebner and Grün APs can substantially reduce the image residuals and refine the accuracy, even if the regular grid patterns are not satisfied. This is a source of the criticism as “have no foundations based on observable physical phenomena” (CLARKE AND FRYER, 1998). However, this confusion can be clarified easily by using the theory of function approximation. The mathematical principle behind Ebner and Grün APs is Weierstrass theorem as well, exactly the same foundation of Legendre APs. Therefore, the irregular patterns only affect the correlations among APs rather than the effect in compensating lens distortion. It is also easy to understand from the approximation view why Ebner APs sometimes achieve quite poor performance. That is, the distortion is too complex for two-order polynomials to well approximate. Higher degree’s polynomials are required and that is why Grün APs perform better in general. Two examples are illustrated for the comparison on the external accuracy in DMC (GSD 20cm) calibration and operational blocks in Fig. 3. It is clear that Ebner APs obtain quite worse accuracy, particularly in the calibration block (Fig. 3 left).

Fig. 3: Comparisons on the external accuracy in DMC (GSD 20cm) calibration block (left) and operational block (right).

However, Legendre APs must be preferred to these two traditional polynomials APs. Firstly, Ebner and Grün APs are single order polynomials while Legendre APs are a whole family of polynomials. Thus, Legendre APs offer much more flexibility for applications. Secondly, from the mathematical viewpoint, continuous polynomials (Legendre APs) are more appropriate than discrete polynomials (Ebner and Grün APs) for digital camera calibration, mainly since the total number of image measurements is much larger than number of unknown APs. Thirdly, Legendre APs are advantageous in low correlations. An example in DMC (GSD 20cm) calibration block is illustrated in Tab. 1, where ‘<0.1’ indicates the percentage of correlations smaller than 0.1 and ‘max’ denotes the maximum correlations. It is demonstrated that Legendre APs have much lower correlations with IO and IMU than Grün APs. The intra-correlations among APs (denoted by ‘intra-corr’) show that Legendre APs are ‘more orthogonal’. Fig. 3 also illustrates that Legendre APs deliver slightly better accuracy than Grün APs. In fact, Legendre APs can also be seen as the superior generalization of the traditional

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<th>IMU</th>
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<td>100%</td>
<td>97%</td>
<td>100%</td>
<td>96%</td>
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<td>max</td>
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<td>0.44</td>
<td>---</td>
<td>0.57</td>
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Tab. 1: Correlation analyses in DMC (GSD 20cm) calibration block
polynomials APs. The discussions were also detailed in Tang et al. (2012).

We also compare Legendre APs with the physical APs proposed in Brown (1971) (Fraser (1997) as well) and Brown (1976), which are denoted by ‘Fraser (10)’ and ‘Brown (21)’ respectively. Although ‘Brown (23)’ model achieves the comparable accuracy as Legendre APs in Fig. 3, the later possesses much better performance in low correlations, as demonstrated in Tab. 1. In fact, the image distortion of the multi-head airborne cameras is not dominated by the radial-symmetric distortion; and this is the main reason why Fraser model delivers rather poor accuracy in Fig. 3 (right). Besides, the mathematical APs hold one inherent advantage, which is again attributed to their mathematical approximation nature. Any physical models, no matter how sophisticated and accurate they might be, can be precisely approximated by the (Legendre) polynomials of proper degree. Although the polynomials APs are sometimes dubbed as ‘empirical’ (McGlove et al., 2004), they are in fact ‘more objective’ in many senses.

4.2 Deficiencies of the Polynomials APs

The solid mathematical principle of Legendre APs (as well as Ebner and Grün APs) is algebraic polynomials approximation. The high performance of Legendre APs has been demonstrated. Generally speaking, Legendre APs are orthogonal, flexible, generic and effective for all the airborne frame-format cameras calibration.

As mentioned in Section 2.1, algebraic polynomials are just one of mathematical basis functions for approximation and can thus serve for calibration purpose. Nevertheless, they may be not the optimal choice. In fact, there is an intrinsic deficiency of all polynomials APs, which is detailed as follows.

It is known that all polynomials APs need to eliminate four high correlated parameters (NOT six, since two of them, i.e. two constant terms, are purely principle point offsets). The elimination imposes four constraints on self-calibrating APs. For example, the constraints on Legendre APs are (see Eq. (13) or (14)),

\[(\Delta x) : + a_1 p_{1,0} + a_2 p_{0,1} + a_3 p_{2,0} + a_4 p_{1,1}
\]

\[(\Delta y) : + a_1 p_{1,0} - a_2 p_{0,1} - a_3 p_{1,1} - a_4 p_{0,2}
\]

The same occurs to the Ebner and Grün APs (see Appendix B). However, these four constraints, which are caused by high correlations, violate the mathematical principle of polynomials APs. According to approximation theory and Weierstrass Theorem, all APs in \(\Delta x\) should be fully independent on the APs in \(\Delta y\). In other words, for 5-order Legendre APs (Ebner and Grün APs), the theoretical number of unknown APs should be 70 (16 and 48) rather than 66 (12 and 44). Therefore, the elimination, which is an indispensable procedure for building all polynomials APs, degrades their rigorousness. This is the intrinsic deficiency of all polynomials APs. Although the negative effect of this deficiency seems insignificant in all our tests, the harmful effect of the deviation from the theory deserves further exploring. Besides, another practical inconvenience is that Legendre APs usually require more unknowns than the physical APs, such as extended Brown models with 21 parameters.

It must be noted that these deficiencies does not imply the failure of the idea of mathematical approximation. Instead, it indicates merely that algebraic polynomials may not be the most proper mathematical basis functions for self-calibration. We shall continue seeking the alternative mathematical basis functions for rigorous self-calibration, which are more justifiable in theory and more efficient in practice.

5 Conclusions

We proposed a new class of self-calibration Legendre APs for calibrating digital frame-format airborne cameras. The prime theoretical foundations of Legendre APs are mathematical polynomial approximation and the renowned Weierstrass Theorem. Theoretically, Legendre APs of proper degree can calibrate the image distortion of all frame cameras.

The high performance of Legendre APs is demonstrated in the many tests on various airborne cameras, including DMC, DigiCam, UltracamX and UltracamXp. Legendre APs are generally
effective and flexible for calibrating all digital frame airborne camera architectures, no matter which system design have been chosen by the camera manufacturer. Moreover, the very low correlations between Legendre APs and other correction parameters guarantee reliable calibration results.

We also compare Legendre APs with other traditional APs. Both the theoretical investigations and the practical experiments show Legendre APs are superior to the conventional Ebner and Grün APs. Compared with the physical APs, Legendre APs show its advantages in general effectiveness and very low correlations. In spite of the encouraging results of Legendre APs, however, we reveal one intrinsic deficiency of all polynomials APs, including those APs in Ebner (1976), Grün (1978) and Legendre APs. It is suggested that further work shall be continued to seek the optimal mathematical choice for rigorous, flexible and efficient self-calibration.

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Appendices

Appendix A: Orthogonal Legendre polynomials

\[ L_0(x) = 1 \]
\[ L_1(x) = x \]
\[ L_2(x) = \frac{1}{2} (3x^2 - 1) \]
\[ L_3(x) = \frac{1}{2} (5x^3 - 3x) \]
\[ L_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3) \]
\[ L_5(x) = \frac{1}{8} (63x^5 - 70x^3 + 15x) \]
\[ L_6(x) = \frac{1}{16} (231x^6 - 315x^4 + 105x^2 - 5) \]

Appendix B: Two conventional polynomials APs
Polynomials APs proposed by Ebner (1976)
\[ \Delta x_e = a_1 x + a_2 y - a_3 2k + a_4 xy + a_5 l + a_6 xl + a_7 yk + a_8 kl \]
\[ \Delta y_e = -a_1 y + a_2 x + a_3 xy - 2a_4 l + a_5 k + a_6 yk + a_7 xl + a_8 kl \]
\[ k = x^2 - 2/3b^2, \quad l = y^2 - 2/3b^2 \]

Polynomials APs proposed by Grün (1978)
\[
\Delta x = a_1 x + a_2 y + a_{xy} - a_1 l - 10 / 7 a_1 k + a_2 x p + a_3 y q + a_4 r + a_5 x y p + a_6 k l \\
+ a_{xy} x p a + a_5 y + a_6 x p + a_7 x y p + a_8 k l \\
+ a_{xy} x p a + a_5 y + a_6 x p + a_7 x y p + a_8 k l \\
+ a_{xy} x p a + a_5 y + a_6 x p + a_7 x y p + a_8 k l \\
= -a_1 y + a_1 x + a_1 / 10 / 7 l - a_1 k + a_2 x y + a_3 y q + a_4 r + a_5 x y p + a_6 k l \\
+ a_{xy} x p a + a_5 y + a_6 x p + a_7 x y p + a_8 k l \\
+ a_{xy} x p a + a_5 y + a_6 x p + a_7 x y p + a_8 k l \\
\]

\( k = x^2 - 1 / 2b^2, \quad l = y^2 - 1 / 2b^2, \quad p = x^2 - 17 / 20b^2, \quad l = y^2 - 17 / 20b^2 \)

\( r^2 = x^2 \left( x^2 - 31 / 28b^2 \right) + 9 / 70b^2 \quad s^2 = y^2 \left( y^2 - 31 / 28b^2 \right) + 9 / 70b^2 \)

**Appendix C: Block Configuration Overview**

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<th>Tab. C.1 Test Block Configuration</th>
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<tr>
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<td>Ultracam-X (GSD 20cm)</td>
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<tr>
<td><strong>DMC (GSD 8cm)</strong></td>
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<td><strong>DMC (GSD 8cm)</strong></td>
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